ON THE MULTIPOINT MIXED FINITE VOLUME METHODS ON QUADRILATERAL GRIDS

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ABSTRACT

In this work, the multipoint flux approximation method is revisited as a mixed finite volume approach. The developments are given for general quadrangular element shape and the theory is given in the physical and the reference spaces. Different spaces for the pressure approximation are studied. Since the well known traditional multipoint flux approximation lead to an unsymmetrical matrix, we propose a new approach which lead to symmetric matrix in the physical space. In the reference space, the both method can be seen as the same numerical approximation. Due to the non conservative form of the inner product integral in the reference element, quadrature rules are introduced to perform method and get close to the physical approximation. Connections are established with the broken Raviart-Thomas mixed finite element given in [Klausen, 2005] and the Brezzi-Douglas-Marini mixed finite element method given in [Wheeler and Yotov, 2005a, Wheeler and Yotov, 2005b].

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbf{R}^2$ with the boundary $\partial\Omega = \Gamma_d \cup \Gamma_n$. We consider the following set of equations,

$$\mathbf{u} = -\mathbf{K}\nabla p \quad \text{on} \quad \Omega, \tag{1}$$

$$s\partial_t p + \nabla.\mathbf{u} = f \quad \text{on} \quad \Omega, \tag{2}$$

$$p = g_d \quad \text{onto} \quad \Gamma_d, \tag{3}$$

$$\mathbf{u}.\mathbf{n} = g_n \quad \text{onto} \quad \Gamma_n. \tag{4}$$

where $\mathbf{u}$ is the flux of the associated state variable p. Equation (2) states for the conservation principle and (1) states the constitutive law like Fourier's law (p is the temperature), Fick's law (p is the concentration of a solute), Ohm's law (p is the electric potential) or Darcy's law (p is the hydraulic head). In this work, we consider the previous system describes the flow through an heterogeneous porous media. $\mathbf{K}$ is a symmetric positive definite permeability tensor, s is the specific storativity in the porous material and f the sink/source term. The associated boundary conditions are of Dirichlet or Neumann type.

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The multipoint flux approximation (MPFA) introduced in [Aavatsmark, 2002] is well suited to solve the previous system. This method is locally mass conservative and has one unknown per element which is the pressure at the cell-center. MPFA can be used for discontinuous and anisotropic media using an unstructured grid [Aavatsmark, 2002, Klausen, 2005, Mischev, 2002]. MPFA can be seen as an appropriate control volume method using a primal and dual mesh [Aavatsmark, 2002]. The dual mesh is constructed by splitting the original element. Assuming continuous fluxes and pressure across each internal edge of a dual mesh element, fluxes could be expressed as a combination of local parameters called transmissibility coefficients. The MPFA method is available on general grid but the final matrix equation is globally non-symmetric and may be non-positive definite for skewed grid and high heterogeneity jump.
In the present paper, we investigate the reformulation of the MPFA method on the mixed form and introduce the multipoint mixed finite volume (MPMFV) method as done in [Mischev, 2002]. The pressure and velocity variables are approximated in different spaces. Different spaces for the pressure approximation are investigated including the traditional MPFA method and the new formulation which always lead to a symmetric matrix in the physical space introduced in [Le Potier, 2005]. In the reference space, these formulations could be seen as a same numerical approximation and quadrature rules are introduced to get close to the physical approximation. Relationship is also discussed with mixed finite element method using different velocity approximation spaces [Aavatsmark et al., 2005, Klausen, 2005].

2. THE MULTIPOINT MIXED FINITE VOLUME METHOD IN THE PHYSICAL SPACE

Let $\mathcal{T}_h$ be a partition of the physical domain Ω into convex quadrangular elements A where h refers the maximal diameter spacing. We assume that each interior vertex is connected with four cells. We denote ∂A the boundary of A and $|A|$ its area . An element of the dual mesh called *interaction region* I is constructed by joining the four subcells with a common corner. The multipoint mixed finite volume formulation assume local variation of the pressure and velocity over each subcell [Mischev, 2002]. The weak formulation of the Darcy law over an element A is given by,

$$(\mathbf{K}^{-1}\mathbf{u}, \chi)_k = -(\nabla p, \chi)_k, \quad \text{with} \quad k = 1..4, \tag{5}$$

where $(.,.)_k$ is the inner product integral defined over the subcell A_k and $|A_k|$ its area. The velocity $\mathbf{u}$ is approximated by a constant $\mathbf{u}_k$ over the subcell respecting the continuous normal components. Different spaces for the pressure approximation are introduced to obtain the weak gradient expression based on linear or constant variation over each subcell. Constant test functions are introduced to discretize both equations. Velocity test function χ is constant over A_k with $\chi(\mathbf{r}) = 1/|A_k|$ if $\mathbf{r} \in A_k$. In the following, we describe the multipoint mixed finite volume procedure in the physical space using the notations in figure 1.
Let $\mathbf{r}_0$ the cell center of the element A, $\bar{\mathbf{r}}_i$ the midpoints of the four boundary edges and $\mathbf{r}_i$ the vertex of the element. Each subcell A_k has the discrete points $\mathbf{r}_0$, $\bar{\mathbf{r}}_k$, $\mathbf{r}_k$ and $\bar{\mathbf{r}}_{k+1}$ for $k = 1..4$ (we assume $\bar{\mathbf{r}}_5 = \bar{\mathbf{r}}_1$). We use a local notation where the external and the inner subedges of the subcell A_k are noted e_{kl} and e_k^l for $k = 1..4$ and $l = 1..2$. The external

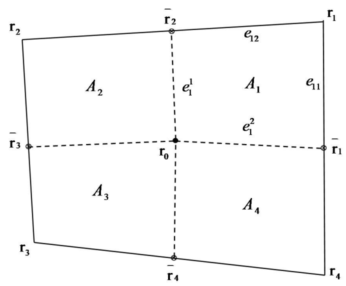

FIGURE 1. A quadrangular cell A and the four subcells A_k.

and inner normal vectors are noted $\mathbf{n}_{kl}$ and $\mathbf{n}_k^l$ respectively. Recall that the inner normal vectors are defined by $\mathbf{n}_k^1 = -(\bar{\mathbf{r}}_{k+1} - \mathbf{r}_o)^\perp$ and $\mathbf{n}_k^2 = (\bar{\mathbf{r}}_k - \mathbf{r}_o)^\perp$.

2.1. **The weak pressure gradient .** By using the Green theorem, the left side of the Darcy's equation could be expressed by,

$$-(\nabla p, \chi)_k = - < p.\bar{\mathbf{n}}, \chi >_k, \tag{6}$$

where $< .,. >_k$ denote a inner product integral defined over the subcell boundary.
First we assume linear pressure variation over each subcell with degrees of freedom $\mathbf{r}_0$, $\bar{\mathbf{r}}_k$ and $\bar{\mathbf{r}}_{k+1}$. We denote this support S_T and its area $|S_T|$. Its gradient is assumed to be constant [Aavatsmark, 2002] and hence we deduce the following decomposition,

$$-(\nabla p, \chi)_k = \frac{1}{2|S_T|} \sum_l^2 \mathbf{n}_k^l.(p_o - \bar{p}_{k-l+1}). \tag{7}$$

Second we assume constant pressure p_o over the subcell and internal subedges. We denote $\bar{p}_k$ and $\bar{p}_{k+1}$ on the external subedges. Hence, the weak gradient expression is given by [Le Potier, 2005],

$$-(\nabla p, \chi)_k = \frac{1}{|A_k|} \sum_l^2 \mathbf{n}_{kl}.(p_o - \bar{p}_{k-l+1}). \tag{8}$$

Which can be written by,

$$-(\nabla p, \chi)_k = \mathbf{X}_k \mathbf{b}_k, \tag{9}$$

where $\mathbf{X}_k$ is a local matrix which contain normal vectors $\mathbf{n}_k^l$ or $\mathbf{n}_{kl}$ depending on the choice of pressure approximation. $\mathbf{b}_k$ corresponds to the pressure difference between the midpoint $\mathbf{r}_0$ and the edge points $\bar{\mathbf{r}}_l$ with $l = k, k+1$.

2.2. **Discrete Darcy law.** The right side of the Darcy's equation is given by,

$$(\mathbf{K}^{-1}\mathbf{u}, \chi)_k = \mathbf{K}^{-1}\mathbf{u}_k. \tag{10}$$

The normal velocity components are deduced by projection on the external normal e_{kl}, $\mathbf{q}_k = \mathbf{Y}_k \mathbf{u}_k$ where $\mathbf{Y}_k$ is a matrix composed of the normal vectors $\mathbf{n}_{kl}$. Coupling the both relations, give the discrete form of the Darcy's law,

$$\mathbf{G}_k.\mathbf{q}_k = \mathbf{b}_k, \tag{11}$$

where $\mathbf{G}_k$ is a 2×2 local matrix.
Finally, we obtain a diagonal block matrix equation over each cell A connecting partial fluxes u_{kl}, mean pressure p_o and pressure at edges (traces pressure) $\bar{p}_k$.

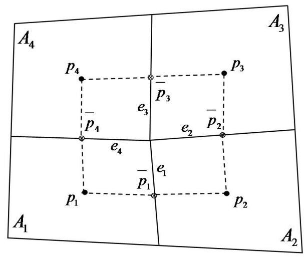

FIGURE 2. An interaction region I constituting by the four subcell A_i where $\bullet$ correspond to the mean pressure evaluation p_i and $\otimes$ the mean pressure evaluation $\bar{p}_i$ on the inner edge of I.

2.3. **Cell-centered stencil.** In this section we expose the procedure to substitute these additional unknowns (traces pressure) and reduce to a cell-centered finite difference system as given in [Klausen, 2005].
We use the interaction region defined in figure 2. We note $\mathbf{G^i_k}$ the one–block matrix related to the subcell A_i of the interaction region and u_i the flux through the subedge e_i.
For example, for adjacent subcells to e_1, we have,

$$A_1 \quad : \quad g^1_{11}u_1 + g^1_{12}u_4 \quad = \quad p_1 - \bar{p}_1, \tag{12}$$

$$A_2 \quad : \quad g^2_{11}u_1 + g^2_{12}u_2 \quad = \quad \bar{p}_1 - p_2 \tag{13}$$

Combining previous equations lead to,

$$(g^1_{11} + g^2_{11})u_1 + g^2_{12}u_2 + g^1_{12}u_4 = p_1 - p_2 \tag{14}$$

Similar equations are obtained for the others subedges $e_2..e_4$.
Finally, we obtained a linear invertible system between fluxes and mean pressures,

$$\mathbf{T}.\mathbf{q} = \mathbf{b} \tag{15}$$

where $\mathbf{q} = [u_1, u_2, u_3, u_4]^T$,

$$\mathbf{T} = \begin{pmatrix} g^1_{11} + g^2_{11} & g^2_{12} & 0 & g^1_{12} \\ g^2_{21} & g^2_{22} + g^3_{22} & g^3_{21} & 0 \\ 0 & g^3_{12} & g^1_{11} + g^2_{11} & g^4_{12} \\ g^1_{21} & 0 & g^4_{21} & g^1_{22} + g^4_{22} \end{pmatrix}$$

The multipoint mixed finite volume method yields to an approximation of the flow problem in the form, $\mathbf{Mp} = \mathbf{f}$ where $\mathbf{M}$ is the final matrix, $\mathbf{p}$ and $\mathbf{f}$ are vectors containing mean pressures, boundaries conditions and sink/source terms. As shown in [Aavatsmark, 2002], symmetry of $\mathbf{M}$ is guarantied only if this property is respected for each local matrix $\mathbf{G}^i_k$ of the interaction region,

$$\text{Sym.}\mathbf{G}^i_k \rightleftharpoons \text{Sym.}\mathbf{T} \rightleftharpoons \text{Sym.}\mathbf{M}$$

Except some particular cases, a linear pressure variation always lead to a non-symmetric mass matrix [Aavatsmark, 2002]. However, by using constant pressure variation, the symmetry is always guarantied independently of the element shape [Le Potier, 2005].

3. THE MULTIPOINT MIXED FINITE VOLUME METHOD IN THE REFERENCE SPACE

The multipoint mixed finite volume method is now discussed in the reference space as done in [Aavatsmark, 2002, Aavatsmark et al., 2005]. Each physical element A could be transformed to the reference element $\hat{A}$ by a bilinear mapping F_A. The Jacobian matrix $\mathbf{J}_A$ and its determinant $|J_A|$ are linearly dependant of $\hat{x}$ and $\hat{y}$. On the reference space, the normal velocity components are preserved by the mapping,

$$u_e = -\int_e \mathbf{K}\nabla p.\bar{\mathbf{n}} = -\int_{\hat{e}} \mathcal{K}\hat{\nabla}\hat{p}.\hat{\mathbf{n}} = \hat{u}_{\hat{e}}, \tag{16}$$

where the tensor $\mathcal{K}$ is defined by $\mathcal{K} = |J_A|\mathbf{J}_A^{-1}.\mathbf{K}.\mathbf{J}_A^{-T}$.
In the reference element $\hat{A}$, the Darcy law is now replaced by $\hat{\mathbf{u}} = -\mathcal{K}\hat{\nabla}p$. Applying the multipoint mixed finite volume approach with both constant and linear pressure approximations lead to the unique matrix notation for the normal velocity,

$$\hat{\mathbf{G}}_k.\hat{\mathbf{q}}_k = \hat{\mathbf{b}}_k \quad \text{where} \quad \hat{\mathbf{G}}_k = \int_{\hat{A}_k} \mathcal{K}^{-1}(\mathbf{r}) = \bar{\mathbf{J}}_k\mathbf{K}^{-1}\bar{\mathbf{J}}_k^T/|\bar{J}_k|. \tag{17}$$

where $\bar{\mathbf{J}}_k$ is a mean Jacobian matrix transformation and $|\bar{J}_k|$ its determinant. Due to the symmetry of $\hat{\mathbf{G}}_k$, the reference method leads always to a symmetric final matrix. Therefore, in the reference space, the multipoint mixed finite volume approach leads to the same system for both constant and linear pressure approximations. However, as stated by [Aavatsmark, 2002, Klausen, 2005], a rupture is occurred between the physical and reference methods due to the non–conservative aspect of the local integral operator, $(\mathbf{K}^{-1}\mathbf{u}, .)_k \neq (\mathcal{K}^{-1}\hat{\mathbf{u}}, .)_k$. For parallelogram elements, these integrals are performed exactly and the equivalence is recovered.

3.1. **Connections to the physical method.** We propose a quadrature rule using discrete Gauss points which correspond to the physical approximation. This study is based on the geometric transformation due to different Jacobian matrix evaluation like midpoint, corner-point values (Figure 3). As explained in [Aavatsmark et al., 2005], the midpoint Jacobian evaluation $\mathbf{J}_o$ conserve exactly the inner subedges but introduces a relative important error in length and orientation of the external subedges and vice-versa for the corner point evaluation $\mathbf{J}_k$. The mean value $\bar{\mathbf{J}}_k$ can be seen as an intermediate matrix.
The main idea is coupling appropriate quadrature rules in order to approximate efficiently the transformed gradient pressure and velocity variables. We define a quadrature rule $\mathcal{L}_g(. \ , \ .)$ combining discrete Gauss point approximation of the pressure gradient, midpoints and corner points approximations for the velocity variable,

$$\mathcal{L}_g(\mathcal{K}^{-1}\hat{\mathbf{u}}, \hat{\chi}) = \bar{\mathbf{G}}_g.\hat{\mathbf{u}}_k \quad \text{where} \quad \bar{\mathbf{G}}_g = \frac{|J_o| + |J_g|}{2|J_g||J_k|}\mathbf{J}_g^T\mathbf{K}^{-1}\mathbf{J}_k, \tag{18}$$

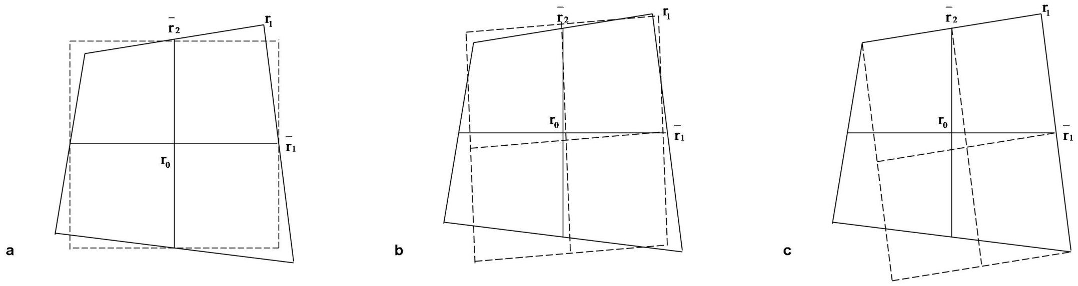

FIGURE 3. The transformation of the quadrilateral subcell A_1 into an approximative parallelogram which is based on midpoint (a), mean integrate (b) and corner-point (c) Jacobian evaluation

Projecting the velocity vector, we obtain the discrete Darcy law,

$$\bar{\mathbf{G}}_g.\hat{\mathbf{q}}_k = \hat{\mathbf{b}}_k, \tag{19}$$

where $\bar{\mathbf{G}}_g$ is not necessary a symmetric mass matrix since $\mathbf{J}_g$ can be different from $\mathbf{J}_k$ depending on the Gauss point evaluation. An easy calculation shows that reference method using midpoint (resp. corner-point) quadrature is identical to the physical method using a linear (resp. constant) pressure variation.

3.2. **Connections to the mixed finite element method.** We introduce another quadrature rule based on Gauss point evaluation of the transformed permeability tensor.

$$\mathcal{L}_g(\mathcal{K}^{-1}\hat{\mathbf{u}}, \hat{\chi}) = \bar{\mathbf{G}}_g\hat{\mathbf{u}}_k \quad \text{where} \quad \bar{\mathbf{G}}_g = \mathcal{K}^{-1}(\mathbf{r}_g), \tag{20}$$

This quadrature approximation of the mixed finite volume method corresponds exactly to the MPFA method using discrete Gauss point for the Jacobian evaluation in the reference space [Aavatsmark, 2002, Klausen, 2005]. Recently, mixed finite element method using broken Raviart-Thomas $\mathcal{RT}_0^{1/2}$ or Brezzi-Douglas-Marini $\mathcal{BDM}_1$ space have been introduced by the authors [Klausen, 2005, Wheeler and Yotov, 2005a]. A coupling with a specific trapezoidal quadrature rule reduce velocity matrix to a diagonal one which lead to a symmetric final matrix. Connections have been clearly established between the reference MPFA method using midpoint and corner point Jacobian evaluation and mixed finite element method in [Aavatsmark et al., 2005, Klausen, 2005].

4. NUMERICAL EXPERIMENTS

Convergence properties are studied for the multipoint mixed finite volume method and compared to the traditional mixed hybrid finite element (MHFE) method using lowest order Raviart-Thomas space. Pressure and normal velocity errors are computed in the discrete $\mathcal{L}_2$–norm. Symmetric or non-symmetric final matrices are solved by iterative solvers using *Preconditioned Conjugate Gradient* (PGC) or *Bi-Conjugate Gradient Stabilized* (Bi-CGS) method, respectively. The Numerical experiments concerned the multipoint mixed finite volume using linear (MPFA) or constant (MPFV) pressure approximation in the physical space. In reference space, we use midpoint and corner point evaluations which correspond to the broken Raviart-Thomas (MFE$_m$) and Brezzi-Douglas-Marini (MFE$_c$) mixed finite element method. We consider the analytical pressure solution defined in the unit square domain Ω by $p(x, y) = \exp[20\pi((x - 1/2)^2 + (y - 1/2)^2)]$ and continuous permeability tensor given by,

$$\mathbf{K} = \begin{bmatrix} y^2 + \alpha x^2 & (\alpha - 1)xy \\ (\alpha - 1)xy & x^2 + \alpha y^2 \end{bmatrix}, \quad \alpha \in \mathbf{R}.$$

Dirichlet boundary conditions and source terms have been implemented to respect the mathematical solution. The multipoint mixed methods are tested for skewed meshes which are generated as an h^γ–perturbed uniform orthogonal meshes (Figure 4.a). The anisotropic criteria is controlled by the parameter value α. Numerical experiments are summarized in the table 1.

5. CONCLUSION

Traditional MPFA method have been described as a particular case of a more general approach which is called multipoint mixed finite volume method. In the physical space, a modification of the pressure variation leads to a new scheme which give a symmetric matrix. In the reference space, appropriate quadrature rules have been defined establishing connections with physical methods. Numerical experiments confirmed that MPFA method cannot be convergent for relatively hard problem (Table 1) but in general it seems to be the more accurate method. The MHFE method is also efficient for smooth problem but for high anisotropic ratio it occurred spurious oscillations cause by the non-monotonicity of the inverse final matrix (Figure 4.c). The MFE_m gives a good pressure approximation but introduce a large velocity error for high distortion grids (Table 1). In this study, MPFV and MFE_c are the most robust methods. Superiority of these symmetric multipoint mixed methods have been clearly established with an optimal pressure and velocity error behaviours and guarantying the use of the PGC algorithm which is the faster iterative solver.

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γ	α	MPFA e_p	MPFA e_v	MPFV e_p	MPFV e_v	MFE$_c$ e_p	MFE$_c$ e_v	MFE$_m$ e_p	MFE$_m$ e_v	MHFE e_p	MHFE e_v
2.0	10^0	1.3E-04	3.6E-05	1.3E-04	5.4E-05	1.1E-04	3.6E-05	1.1E-04	1.6E-04	1.1E-04	4.3E-05
	10^2	2.1E-04	3.8E-03	2.1E-04	3.8E-03	2.1E-04	3.8E-03	2.2E-04	1.1E-02	3.3E-03	5.6E-03
	10^3	2.3E-04	3.8E-02	2.3E-04	3.8E-02	2.3E-04	3.8E-02	3.4E-04	1.0E-01	3.6E-02	6.5E-02
1.5	10^0	1.6E-04	4.8E-05	2.4E-04	4.5E-05	8.6E-05	7.5E-05	1.7E-04	1.8E-03	1.1E-03	7.0E-05
	10^2	2.1E-04	4.1E-03	2.5E-04	7.1E-03	2.5E-04	6.0E-03	2.1E-03	1.1E-01	3.3E-03	1.1E-02
	10^3	2.5E-04	4.2E-02	3.2E-04	5.3E-02	3.2E-04	4.7E-02	1.9E-02	1.1E-00	3.6E-02	1.2E-01
1.0	10^0	2.0E-04	2.5E-04	2.1E-03	2.8E-03	2.3E-03	1.2E-03	3.1E-03	1.2E-02	1.3E-04	4.0E-04
	10^2	–	–	1.3E-03	4.4E-02	1.5E-03	4.3E-02	8.8E-02	8.7E-01	4.3E-03	7.1E-02
	10^3	–	–	1.8E-03	2.8E-01	1.8E-03	2.6E-01	–	–	4.6E-02	7.6E-01

TABLE 1. Pressure and normal velocity error for these Mixed methods using h^γ–perturbed grid and different α value.

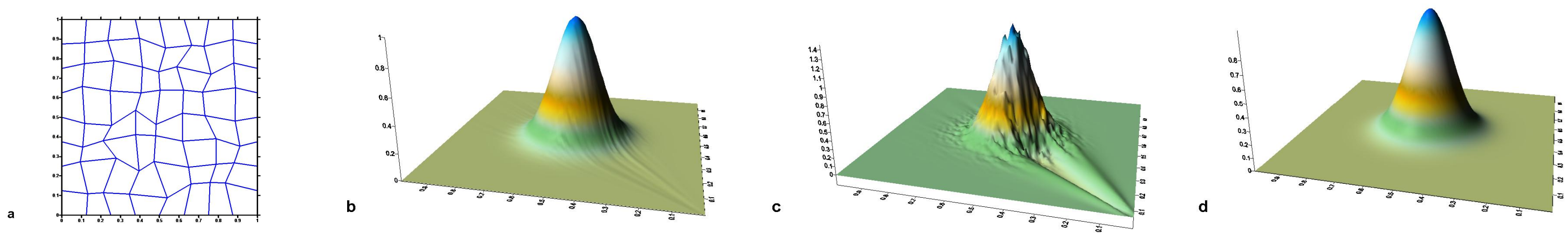

FIGURE 4. The figure (a) represent an h^1–perturbation of a uniform orthogonal grid. Comparing the numerical solution using MPFV (b) and MHFE (c) to the analytical solution method (d) for very hard problem (h^1–perturbed grid and $\alpha = 10^3$)