MODELING FORCHHEIMER FRACTURES AS INTERFACES

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Abstract

In this article we are concerned with modeling single phase flow in a medium with known fractures. In particular we are interested in the case in which the flow rate in the fractures is large enough to make it appropriate to use Forchheimer's law for modeling the flow in the fractures even though the flow in the surrounding domain is such that Darcy's law is adequate. We introduce a model in which the fractures are treated as interfaces.

1. INTRODUCTION

Modeling flow and transport of contaminants in porous media is made difficult by the presence of heterogenieties in the characteristics of the medium occurring at scales quite different from those describing the average characteristics of the meduim. One particular instance of this phenomena is the occurrence of fractures in the meduim, regions very small in width but very important for modeling flow because of their much higher (or possibly much lower) permeability. Fine networks of interconnected fractures occuring with some degree of regularity are often taken into account by double porosity models [5]. Here however we are concerned with larger fractures or faults of known location that need to be included specifically in the model. Some models for networks of such fractures ignore any interaction between the fractures and the surrounding rock [6] . We would like to have a model for these fractures that takes into account interaction between the fractures and the matrix as do double porosity models. In earlier works, [1] and [7], a model was introduced in which the fracture was treated as a lower dimensional domain, as an interface between two subdomains. At these fracture interfaces, the flux continuity condition was replaced by an equation representing Darcy flow along the interface.

When the flow in the fracture is sufficiently rapid however, inertial effects need to be taken into account, and Forchheimer's law describes the flow in the fracture more accurately than does Darcy's law.

The object of this article is to extend the above model to the case in which the flow along the fracture is governed by Forchheimer's law. As Forchheimer's law is given by

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a nonlinear equation, this model is more complex. It is derived by averaging across the fracture under the assumption that the flow in the direction normal to the fracture is less important than in the directions tangential to the fracture. Domain decomposition type techniques can still be used as the nonlinearities involve only the interface unknowns. A quasi-Newton method can then be used to solve the resulting nonlinear problem on the interfaces. In this article the numerical model will be derived and numerical results will be presented.

In Section 2, we define the laws controlling the flow in a porous media. In Section 3, we describe the problem of flow in a domain containing a fracture and in Section 4, the problem model is derived. In Section 5, we obtain the discrete problem model and in Section 6 we give numerical results, and we will finish with a conclusion in Section 7.

2. THE EQUATIONS GOVERNING FLOW IN A POROUS MEDIUM

Flow of an incompressible fluid in a porous medium Ω is governed by the law of mass conservation:

$$\text{div}\,(\mathbf{u}) \;=\; q \quad \text{in} \quad \Omega, \tag{1}$$

where q is a source term and $\mathbf{u}$ is a volumetric flow rate or velocity. The velocity $\mathbf{u}$ is related to the gradient of the pressure p (in the absence of gravity) by Darcy's law:

$$\mathbf{u} \,+\, K\,\nabla\,p \;=\; 0, \quad \text{in} \quad \Omega, \tag{2}$$

where K is the permeability of the medium.

Darcy's law is valid for low flow rates for which inertial effects are negligible. For higher flow rates however, the inertial effects are more important and Forchheimer's law gives a more accurate relation between the gradient of the pressure and the flow rate; cf. [**?**, AGR]r [**?**, Dgls] Forchheimer's law is a nonlinear law given by

$$G(\mathbf{u}) + K\,\nabla p \;\;=\;\; 0 \quad \text{in} \quad \Omega, \tag{3}$$

with G is defined by $G(\mathbf{u}) \;=\; (1 + b\,|\mathbf{u}|)\,\mathbf{u}$, with b is a constant of inertia.

3. DESCRIPTION OF A SIMPLE MODEL PROBLEM

For this simple model problem we suppose that Ω is a convex domain in $\mathbb{R}^n$, $n = 2$ or 3, containing a single fracture Ω_f. We suppose (see Figure 1) that the subdomain Ω_f is such that there is a hyperplane γ with unit normal vector $\mathbf{n}$ so that

$$\begin{aligned}\Omega_f \;\;=\;\; \{x \in \Omega \,:\, x = s + r\mathbf{n} \text{ for some } s \in \gamma \cap \Omega \\ \text{and some } r \text{ in the interval}(-\tfrac{d(s)}{2}, \tfrac{d(s)}{2})\},\end{aligned}$$

where $d(s)$ denotes the thickness of the fracture at $s \in \gamma$. We also suppose that $\overline{\Omega}_f$ separates Ω into two disjoint, connected subdomains:

$$\Omega \backslash \overline{\Omega}_f = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \emptyset.$$

We will use the notation $\Gamma = \partial\Omega$ for the boundary of Ω and $\Gamma_i = \partial\Omega_i \cap \Gamma$ for the part of the boundary of Ω_i, which lies on Gamma, $i = 1, 2$.

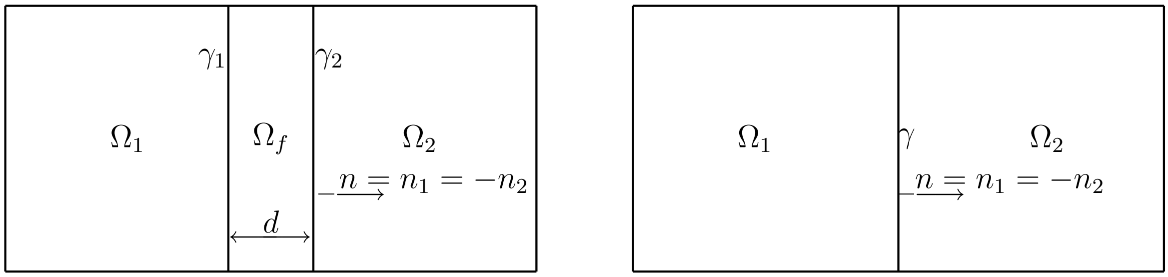

FIGURE 1. Left: the domain Ω with the fracture Ω_f. Right: the subdomains Ω_1 and Ω_2 separated by the fracture considered as an interface γ.

We suppose that the flow in Ω is governed by a conservation equation together with Forchheimer's law relating the gradient of the pressure p to the flow velocity $\mathbf{u}$:

$$\begin{aligned} \operatorname{div}\mathbf{u} &= q && \text{in } \Omega \\ (1 + b|\mathbf{u}|)\mathbf{u} &= -K\nabla p && \text{in } \Omega \\ p &= p_D && \text{on } \Gamma, \end{aligned} \tag{4}$$

where q is a source term and p_D is the given pressure on the boundary Γ. We suppose that the permeability (or hydraulic conductivity) K is positive and bounded above and away from 0:

$$0 < K_{min} \leq K \leq K_{max} < \infty,$$

and the coefficient of inertia b is nonnegative.

The case of interest for us is that the velocity in the subdomains is small enough for Darcy's law to be sufficient while that in the fracture requires the use of Forchheimer's law. In this case in order to avoid having to solve a nonlinear problem in all of Ω it would seem appropriate to formulate the problem as a transmission problem using the nonlinear law only in the fracture. If we assume that the flow in the subdomains is governed by a Darcy's law and if we denote by $p_i, \mathbf{u}_i, K_i,$ and q_i the restrictions of $p, \mathbf{u}, K,$ and q respectively to $\Omega_i,\ i = 1, 2, f,$ and by p_{i_D} the restriction of p_D to $\Gamma_i,\ i = 1, 2, f$ we can rewrite the above problem (4) as a transmission problem:

$$\begin{aligned} \operatorname{div} u_i &= q_i && \text{in } \Omega_i, \quad i = 1, 2, f, \\ \mathbf{u}_i &= -K_i \nabla p_i && \text{in } \Omega_i, \quad i = 1, 2, \\ (1 + b|\mathbf{u}_f|)\mathbf{u}_f &= -K_f \nabla p_f && \text{in } \Omega_f, \\ p_i &= p_{i_D} && \text{on } \Gamma_i, \quad i = 1, 2, f, \\ p_i &= p_f && \text{on } \gamma_i, \quad i = 1, 2, \\ \mathbf{u}_i \cdot n &= \mathbf{u}_f \cdot n && \text{on } \gamma_i, \quad i = 1, 2. \end{aligned} \tag{5}$$

The system (5) could then be solved using a conventional nonoverlapping domain decomposition technique in which Ω_f would be considered simply as a third subdomain. The alternative technique proposed here is to treat Ω_f not as a subdomain, but as an interface γ between the subdomains Ω_1 and Ω_2 (Figure 1) and to impose non local transmission conditions .

4. DERIVATION OF THE MODEL

The model in which the subdomain Ω_f is replaced by the interface γ, is obtained by using the technique of averaging across the fracture. For this however, a simplifying hypothesis, that flow in the fracture in the direction normal to the central hyperplane γ is much smaller than that in the tangential direction, is used. This bypotesis is justified by the very small ratio, width to length of the fracture. Thus we suppose that the flow in the direction normal to the fracture is adequately described by Darcy's law.

The first step toward deriving the model is to decompose $\mathbf{u}_f$ as $\mathbf{u}_f = \mathbf{u}_{f,n} + \mathbf{u}_{f,\tau}$ with $\mathbf{u}_{f,n} = (\mathbf{u}_f \cdot \mathbf{n})\,\mathbf{n}$ (recall that that $\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$), and to introduce the notation ∇_τ and div_τ for the tangential gradient and divergence operators and ∇_n and div_n for the normal gradient and divergence operators.

4.1. **Averging the conservation equation:** With the above notation, the first equation of (5) for $i = f$, may be rewritten as

$$\mathrm{div}_n\, \mathbf{u}_f + \mathrm{div}_\tau\, \mathbf{u}_f = q_f \quad \text{in} \quad \Omega_f \tag{6}$$

Integrating in the direction normal to the fracture, one obtains

$$\mathbf{u}_f \cdot \mathbf{n}_{|\gamma 2} - \mathbf{u}_f \cdot \mathbf{n}_{|\gamma 1} + \mathrm{div}_\tau\, \mathbf{U}_f = Q_f \quad \text{on} \quad \gamma \tag{7}$$

where $\mathbf{U}_f = \displaystyle\int_{-\frac{d}{2}}^{\frac{d}{2}} \mathbf{u}_{f,\tau}\, dn$ and $Q_f = \displaystyle\int_{-\frac{d}{2}}^{\frac{d}{2}} q_f\, dn$. Then using the continuity of the fluxes across γ_1 and γ_2, the last equation of (5) for $i = 1$ and 2, we may be write

$$\mathrm{div}_\tau\, \mathbf{U}_f = Q_f + (\mathbf{u}_1 \cdot \mathbf{n}_{1|\gamma 1} + \mathbf{u}_2 \cdot \mathbf{n}_{2|\gamma 2}) \qquad \text{on} \quad \gamma. \tag{8}$$

This is the conservation equation on γ with the additional source term $\mathbf{u}_1 \cdot \mathbf{n}_{1|\gamma 1} + \mathbf{u}_2 \cdot \mathbf{n}_{2|\gamma 2}$.

4.2. **Averging Forchheimer's law:** With the hypothesis that flow in the fracture in the direction normal to the fracture is described by Darcy's law, the decomposition of the second equation of the system (5) into its tangential-direction and the normal-direction parts is given by

$$\begin{array}{rcll} (1 + b\,|\mathbf{u}_f|)\,\mathbf{u}_{f,\tau} & = & -K_{f,\tau}\,\nabla_\tau\, p_f & \text{(a)} \\ \mathbf{u}_{f,n} & = & -K_{f,n}\,\nabla_n\, p_f. & \text{(b)} \end{array} \tag{9}$$

Integrating the first equation of the system (9, a) along the line segments $(-\frac{d}{2}, \frac{d}{2})$, yields:

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} (1 + b\,|\mathbf{u}_f|)\,\mathbf{u}_{f,\tau}\, dn = -K_f\,\nabla_\tau \int_{-\frac{d}{2}}^{\frac{d}{2}} p_f\, dn, \tag{10}$$

where we have assumed that $K_{f,\tau}$ is constant along the segments $(-\frac{d}{2}, \frac{d}{2})$. We approximate $|\mathbf{u}_f| = |\mathbf{u}_{f,\tau} + \mathbf{u}_{f,n}|$ by $|\mathbf{u}_{f,\tau}|(1 + \frac{1}{2}\, \frac{\mathbf{u}_{f,n}^2}{\mathbf{u}_{f,\tau}^2})$. Then using the hypothesis that $\mathbf{u}_{f,n}$ is very small in comparison to $\mathbf{u}_{f,\tau}$, we obtain the approximation

$$|\mathbf{u}_f| \simeq |\mathbf{u}_{f,\tau}| \simeq \frac{|\mathbf{U}_f|}{d}.$$

Replacing $|\mathbf{u}_f|$ by $\frac{|\mathbf{U}_f|}{d}$ in (10) and integrating one obtains

$$(1+\frac{b}{d}|\mathbf{U}_f|\ )\ \mathbf{U}_f = -K_{f,\tau}\ d\ \nabla_\tau P_f. \tag{11}$$

where $P_f = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} p_{f,\tau}\ dn$. This is Forchheimer's law in the $(n-1)$ dimensional domain γ. Together (8) and (11) give a flow equation in γ with a source term representing the flow from the subdomains Ω_1 and Ω_2 into the fracture. The second equation of (9) must now be used to give boundary conditions along γ for the subdomains Ω_1 and Ω_2. With the hypoteses that $\mathbf{u}_{f,n}$ is small and $K_{f,n}$ is large, we suppose that the pressure p is continuous through γ so that

$$p_{1|\gamma} = P_f = p_{2|\gamma}.$$

The interface model for the fracture problem can now be written

$$\begin{array}{rcll} \text{div}\,\mathbf{u}_i &=& q_i & \text{in } \Omega_i, \quad i=1,2 \\ \mathbf{u}_i &=& -K_i \nabla p_i & \text{in } \Omega_i, \quad i=1,2 \\ \text{div}_\tau\,\mathbf{U}_f &=& Q_f + (\mathbf{u}_1\cdot\mathbf{n} - \mathbf{u}_2\cdot\mathbf{n}) & \text{on } \gamma \\ (1+\frac{b}{d}|\mathbf{U}_f|)\,\mathbf{U}_f &=& -K_{f,\tau}\, d\, \nabla_\tau P_f & \text{on } \gamma \\ p_i &=& p_{_D} & \text{on } \Gamma_i, \quad i=1,2 \\ p_i &=& P_f & \text{on } \gamma, \quad i=1,2 \\ P_f &=& P_{f_D} & \text{on } \partial\gamma, \end{array} \tag{12}$$

where P_{f_D} is the average value of $p_{_D}$ along the segment $(-\frac{d}{2},\frac{d}{2})$ in $\partial\gamma$.

5. THE DISCRETE MODEL

To calculate an approximate solution of the system (12), a mixed finite element method was use. Let $\mathcal{T}_h = \cup\mathcal{T}_{h,i}$ be a conforming finte element partition of $\overline{\Omega} = \cup\overline{\Omega}_i$, $i=1,2$ so that the meshes $\mathcal{T}_{h,i}$, $i=1,2$ match on the interface γ. Let then $\mathcal{T}_{h,f}$ denote the induced mesh on γ consisting of all edges of elements of $\mathcal{T}_{h,i}$ that lie on γ.

Let $M_{h,i}$ be the space of functions on Ω_i that are constant one each element T of $\mathcal{T}_{h,i}$, and let $M_{h,f}$ be the space of functions on the interface γ that are constant on each E of $\mathcal{T}_{h,f}$. Then let M_h be the $M_{h,1}\times M_{h,2}\times M_{h,f}$.

The approximation space $\mathbf{W_h}$ for the vector functions will be the product space $\mathbf{W}_{h,1}\times \mathbf{W}_{h,2}\times\mathbf{W_{h,f}}$, where $\mathbf{W_{h,i}}$ is the Raviart-Thomas-Nedelec space of lowest order on Ω_i subordinate to the mesh $\mathcal{T}_{h,i}$, and $\mathbf{W_{h,f}}$ is the Raviart-Thomas-Nedelec space of lowest order on γ subordinate to the mesh $\mathcal{T}_{h,f}$.

The discrete mixed finite element approximation for problem (12) is to find $(\mathbf{u}_{h,1},\mathbf{u}_{h,2},\mathbf{U}_{h,f})\in \mathbf{W}_h$ and $(p_{h,1},p_{h,2},P_{h,f})\in M_h$ such that for all $(\mathbf{v}_{h,1},\mathbf{v}_{h,2},\mathbf{V}_{h,f})\in\mathbf{W}_h$ and $(r_{h,1},r_{h,2},R_{h,f})\in$

M_h one has

$$(\mathcal{P}_h)\quad \begin{aligned} &\int_{\Omega_i} K_i^{-1}\mathbf{u}_{h,i}\cdot\mathbf{v}_{h,i} - \int_{\Omega_i} p_{h,i}\mathrm{div}\mathbf{v}_{h,i} = \int_\gamma P_{h,f}\mathbf{v}_{h,i}\cdot\mathbf{n}_i + \int_{\Gamma_i} p_{i_D}\mathbf{v}_{h,i}\cdot\mathbf{n}_i,\ i=1,2\\ &\int_{\Omega_i}\mathrm{div}\mathbf{u}_{h,i}r_{h,i} = \int_{\Omega_i} q_i r_{h,i},\quad i=1,2\\ &\int_\gamma K_f^{-1}(1+\frac{b}{d}|\mathbf{U}_{h,f}|)\mathbf{U}_{h,f}\cdot\mathbf{V}_{h,f} - \int_\gamma P_{h,f}\mathrm{div}\mathbf{V}_{h,f} = \int_{\gamma\cap\Gamma} P_{f_D}\mathbf{V}_{h,f}\cdot\mathbf{n}_f\\ &\int_\gamma \mathrm{div}_\tau\mathbf{U}_{h,f}R_{h,f} = \int_\gamma Q_f R_{h,f} + \int_\gamma(\mathbf{u}_{h1}\cdot\mathbf{n}_1 + \mathbf{u}_{h,2}\cdot\mathbf{n}_2)R_{h,f}. \end{aligned}$$

To solve problem $(\mathcal{P}_h)$, a domain decomposition type method was used ([8]). First the problem was reduced to a problem on the interface γ using Steklov-Poincare operators. For each i, the Steklov-Poincaré operator

$$\begin{aligned} \mathcal{S}_i : M_{h,f} &\longrightarrow M_{h,f}\\ r_{h,f} &\mapsto -\mathbf{u}^0_{h,i}\cdot n_i \end{aligned}$$

associates to an element $R_{h,f}\in M_{h,f}$ the element $-u^0_{h,i}\cdot\mathbf{n}_i\in M_{h,f}$ where $(\mathbf{u}^0_i, p^0_{h,i})\in \mathbf{W_{h,i}}\times M_{h,i}$ is the solution of the problem

$$(\mathcal{P}^0_i)\quad \begin{aligned} &\int_{\Omega_i} K_i^{-1}\mathbf{u}^0_{h,i}\cdot\mathbf{v}_{h,i} - \int_{\Omega_i} p^0_{h,i}\mathrm{div}\mathbf{v}_{h,i} = \int_\gamma R_{h,f}\mathbf{v}_{h,i}\cdot\mathbf{n}_i \quad \forall\mathbf{v}_{h,i}\in\mathbf{W_{h,i}}\\ &\int_{\Omega_i}\mathrm{div}\mathbf{u}^0_{h,i}r_{h,i} = 0 \qquad \forall r_{h,i}\in M_{h,i}. \end{aligned}$$

Also needed is $\chi_i = \mathbf{u}^*_{h,i}\cdot n_i\ i=1,2$, where $(u^*_{h,i}, p^*_{h,i})\in\mathbf{W_{h,i}}\times M_{h,i}$ is the solution of the problem

$$(\mathcal{P}^*_i)\quad \begin{aligned} &\int_{\Omega_i} K_i^{-1}u^*_{h,i}\cdot\mathbf{v}_{h,i} - \int_{\Omega_i} p^*_{h,i}\mathrm{div}\mathbf{v}_{h,i}\int_{\Gamma_i} p_D\mathbf{v}_{h,i}\cdot\mathbf{n}_i \quad \forall\mathbf{v}_{h,i}\in\mathbf{W_{h,i}}\\ &\int_{\Omega_i}\mathrm{div}\mathbf{u}^*_{h,i}r_{h,i} = \int_{\Omega_i} q_i r_{h,i} \qquad \forall r_{h,i}\in M_{h,i}. \end{aligned}$$

Then if $(p_{h,1}, p_{h,2}, P_{h,f})$ with $(\mathbf{u}_{h,1},\mathbf{u}_{h,2},\mathbf{U}_{h,f})$ is the solution for $(\mathcal{P}_h)$ and $(\mathbf{u}^0_{h,i}, p^0_{h,i})$ is the solution of $(\mathcal{P}^0_i)$ with $R_{h,f} = P_{h,f}$ one has

$$p_{h,i} = p^0_{h,i} + p^*_{h,i} \quad \text{et} \quad \mathbf{u}_{h,i} = \mathbf{u}^0_{h,i} + \mathbf{u}^*_{h,i}. \tag{13}$$

Thus $\mathbf{u}_{h,i}\cdot\mathbf{n}_i = \mathcal{S}_i(P_{h,f}) + \chi_i$, and $(\mathbf{U}_{h,f}, P_{h,f})$ is the solution of

$$(\mathcal{P}_\gamma)\quad \begin{aligned} &\text{Find } (\mathbf{U}_{h,f}, P_{h,f})\in\mathbf{W_{h,f}}\times M_{h,f}\text{such that } \forall R_{h,f}\in M_{h,f} \text{ and } \mathbf{V}_{h,f}\in\mathbf{W_{h,f}},\\ &\int_\gamma K_f^{-1}(1+\frac{b}{d}|\mathbf{U}_{h,f}|)\mathbf{U}_{h,f}\cdot\mathbf{V}_{h,f} - \int_\gamma P_{h,f}\mathrm{div}\mathbf{V}_{h,f} = \int_{\partial\gamma} p_D\mathbf{V}_{h,f}\cdot\mathbf{n}_f\\ &\int_\gamma\mathrm{div}_\tau\mathbf{U}_{h,f}R_{h,f} + \int_\gamma(\mathcal{S}_1(P_{h,f}) + \mathcal{S}_2(P_{h,f}))R_{h,f} = \int_\gamma Q_f R_{h,f} + \int_\gamma(\chi_1+\chi_2)R_{h,f}. \end{aligned}$$

The non-linear problem $(\mathcal{P}_\gamma)$ is solved using a quasi-Newton method. The solutions $(u_{h,i}, p_{h,i})$ in the subdomains can then be calculated using (13).

6. NUMERICAL RESULTS

In this section we compare numerical results obtained with the interface model with those obtained using a standard model with 2D fracture. The results are obtained for a simple model problem: the subdomains Ω_1 and Ω_2 are both squares of unit length and width, and the fracture Ω_f separating the subdomains is of unit length and of width $d = 0.01$. The permeability in each of the subdomains is assumed to be constant and equal to 10^{-9}, and the fracture is assumed to be of much higher (also constant) permeability, $K_f = 10^{-6}$. The Forchheimer coefficient b is taken to be 10. The upper and lower boundaries of the the two subdomains are assumed to be impermeable, and there is a pressure drop from right to left of 10^8. The same pressure drop from top to bottom of the fracture is imposed. See Figure 2.

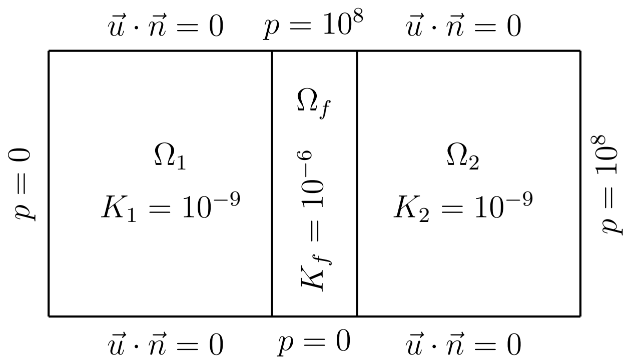

FIGURE 2. Test-case

In Figure 3 the computed pressure field is shown. The result on the left was obtained using the interface model and that on the right is the reference pressure computed with a 2D fracture. In both cases a 40 by 40 grid was used for each of the subdomains Ω_1 and Ω_2, and in the case of the reference model a 40 by 40 grid was also used for the fracture domain Ω_f.

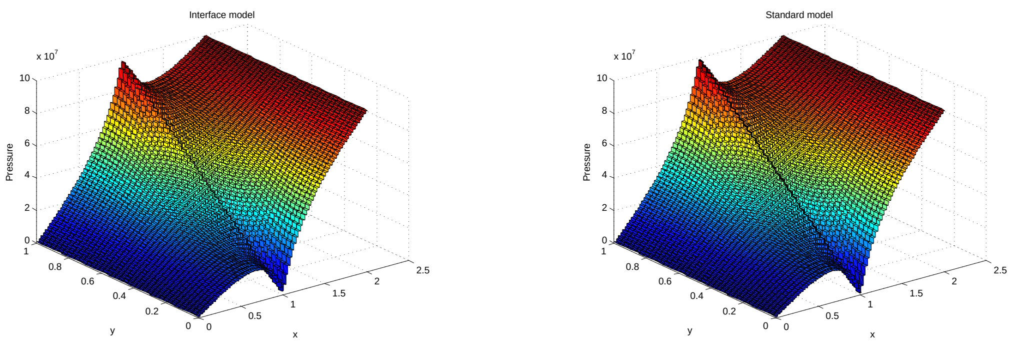

FIGURE 3. Pressure for the interface fracture model (left) and reference pressure (right)

To obtain a quantitative appreciation of the error committed in using the interface model, we have used pressure and velocity fields calculated with the 2D fracture model using a 40 by 40 grid on each subdomain for the discretisation as the reference solution.

We have then calculated the pressure and velocity with the interface fracture model using 3 grids: a 10 by 10 grid for each subdomain, a 20 by 20 grid for each subdomain and a 40 by 40 grid for each subdomain. We then calculated the L^2 error for both the pressure and the velocity . The results are given in the following table.

L^2-Error		
Grid	Pressure	Velocity
40×40	7.9655×10^{-5}	4.4010×10^{-11}
20×20	7.6922×10^{-4}	5.6603×10^{-08}
10×10	18.00×10^{-4}	9.3092×10^{-08}

7. CONCLUSION

The model presented in this paper makes it possible to treat the case of fractures in which the flow is such that the inertial term is not negligible. The numerical results obtained with a very simple model problem are promising. This model still needs to be analyzed theoretically and more realistic numerical experiments in 3D with intersecting fractures need to be realized. It could also be of interest to extend the work here to the case of the model with discontinuous pressure at the interface. Such a model might be appropriate for fractures with high tangential permeability so that Forchheimer's law applies, but with low normal permeability so that the fracture acts to some extent as a barrier.

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