FUZZY KALMAN FILTERING OF HYDRAULIC CONDUCTIVITY

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ABSTRACT

This paper addresses the utilization of expert information in groundwater hydrology. Given that a fuzzy model of hydraulic conductivity can be provided by appropriate experts for some spatial domain, crisp measurements can be used to update the model using a fuzzy Kalman filter. The motivation behind this algorithm is to respect the relative integrity of each data source, while still taking advantage of both forms of data.

1. INTRODUCTION

The characterization of imprecise information in hydrogeological applications has frequently been achieved using fuzzy logic, a non-frequentist methodology; and, in a number of these applications, fuzzy sets are used to define imprecise values of a variable. Too often, though, fuzzy data are assumed to have the same relative reliability as crisp (non-fuzzy) data. The former are generally provided by an appropriate expert, and can be rather imprecise. Such data should be used to provide a starting point for most problems. In hydrogeological applications, crisp data are acquired through in- or ex-situ testing. If an initial fuzzy guess is available, the crisp data should update the fuzzy data, rather than be used in conjunction with it.

Consider the problem of estimating hydraulic conductivity. Such a task is as crucial to groundwater modeling as it is difficult to perform accurately. This difficulty is tied to the high cost identified with groundwater sampling, which often limits the number of hydraulic conductivity measurements at a given site. A small number of crisp measurements, combined with preferential sampling can make estimates, namely kriged estimates, relatively uninformed and unrealistic. Thus, data derived from expert knowledge to supplement the available measured data could be quite valuable in procuring hydraulic conductivity estimates of considerable accuracy.

However, in contrast to methods where fuzzy and crisp data are kriged together to develop a fuzzy hydraulic conductivity model for the spatial domain, the work described herein views fuzzy and crisp data in different lights, utilizing each toward an end respective of its relative reliability. Namely, expert-derived data are used to define a starting point for a hydraulic conductivity model, namely an imprecise idea of what the hydraulic conductivity field is for a site. Available crisp data update this fuzzy model using a fuzzy Kalman filter. Presuming that the fuzzy model is sufficiently accurate, the crisp data fine-tune the model and produce accurate, yet fuzzy, estimates of hydraulic conductivity. An example is provided to illustrate the usefulness of this algorithm.

2. FUZZY LOGIC

Fuzzy logic provides a theory by which non-traditional sources of data can be characterized to facilitate use with more traditional data. Specifically, in this case, expert knowledge is considered to be a non-traditional data type that, without fuzzy logic, is difficult to accurately capture. The fuzzy set is the trademark feature of fuzzy logic and is the mechanism by which imprecise human input is reflected.

A fuzzy set is defined by a membership function, $\mu(x)$, which, for the variable of interest, X, stipulates to what degree certain variable values belong to a given imprecise notion. For instance, it is possible for an appropriate expert to claim with certainty, based upon some evidential motivation, that the hydraulic conductivity at some location is *about 3* millidarcies. The expert is certain but imprecise.

Without fuzzy sets, defining the notion of *about 3* (Figure 1) would be a dubious enterprise. In fact, Figure 1 is an example of a particular type of fuzzy set, called the fuzzy number. A fuzzy number is a fuzzy set that satisfies three criteria [*Klir*, 1995]:

(i) The fuzzy set must have at least one value with full membership (normality);
(ii) α-cuts (see below) are closed intervals for $0 < \alpha \leq 1$;
(iii) The support of the fuzzy set, the zero-level cut, must be bounded.

It should be evident, from the *about 3* example that fuzzy numbers are the imprecise counterpart of traditional (crisp) numbers.

Definitions of fuzzy sets are quite subjective. In scientific applications, only appropriate experts should be petitioned to provide data in the form of fuzzy sets. By utilizing expert knowledge in scientific applications, it is expected that variability in fuzzy set definition, due to subjectivity, is reduced to an acceptable amount.

An important feature of the fuzzy set is the α-cut. Membership values, along the vertical axis in Figure 1, are also referred to as α-values. Horizontally cutting a fuzzy set at a given α-value produces an interval of values, called an α-cut.

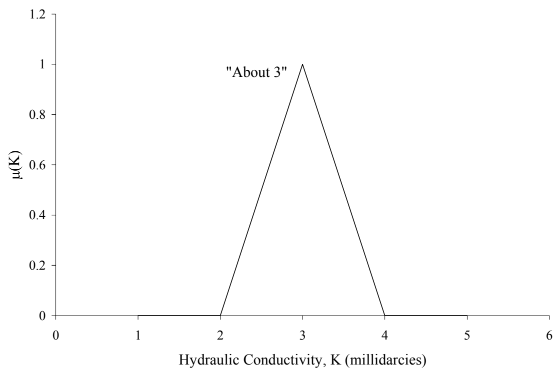

FIGURE 1: One expert's definition of "About 3," defined for hydraulic conductivity

Fuzzy numbers can be operated upon using the same equations that operate on crisp numbers. This is possible through the extension principle [*Zadeh*, 1965]. The extension principle retrofits traditional formulae for use with fuzzy numbers.

3. FUZZY VARIOGRAM

For a given site, knowledgeable hydrogeologists can provide their impression of hydraulic conductivity values as fuzzy numbers at various locations throughout the spatial domain. Geostatistical kriging is used to create estimates at regular intervals throughout the entire domain. As long as the measurements are fuzzy numbers, fuzzy kriging is required. The kriging approach uses weighted sums of available fuzzy "measurements" to calculate fuzzy estimates at un-sampled locations.

The fuzzy kriging method used herein is rather straightforward, and requires only a slight change to the determination of the sample semivariogram [*Bardossy*, 1989], which is a representation of the spatial covariance structure. This slight change is, in fact, the calculation of fuzzy sample semivariance values, rather than traditional crisp values.

The most important step in constructing the sample variogram is the calculation of the squared differences between pairs of measurements. However, one must apply the extension principle in order to find the fuzzy squared difference between a pair of fuzzy measurements.

For each pair of measurements, there is now a calculated squared difference and a known physical distance between their measurement locations. Measurement pairs with sufficiently similar separation distances are grouped together, their squared differences and separation distances averaged. The resulting averaged distance, h, and average fuzzy squared difference, $\gamma(h)$, are plotted to produce the sample variogram (Figure 3). To this sample semivariogram is fit a crisp model semivariogram selected from a prescribed set of possible positive definite models. The significance of this is that the weights calculated by the kriging equations are crisp values. Multiplying the crisp weights by the fuzzy values produces a fuzzy estimate at each desired location in the spatial domain.

4. KALMAN FILTER

The Kalman filter is a series of predicting and updating steps generally employed to estimate the state of a dynamic system, such as the location of some projectile object. The filter, named for Rudolf Kalman, relies upon measurements of the state variable (location, in the projectile object scenario) to perform the updating steps, while an understanding of the processes that govern the dynamics are requisite for prediction.

Typically, the Kalman filter algorithm begins with an educated guess as to what the state of the system is. After acquiring measurements of the state variable, the educated guess is refined (updating step). Employing an understanding of the system dynamics, the state of the system is forecasted for the next time step (prediction step). A new set of measurements incites more updating, and the process ensues until measurements are no longer made [*Welch and Bishop*, 2001].

However, in this particular application, the system is static (no prediction step required), only one updating step is performed for the one set of crisp measurements, and the educated guess that incites the algorithm is the set of fuzzy hydraulic conductivity estimates that result from the fuzzy kriging. Thus crisp measurements are used to update the fuzzy model in this fuzzy Kalman filter.

There has been a significant amount of research into the use of fuzzy logic in Kalman filtering [*Chen et al.*, 1998; *Chiang*, 2003]. However, no evident research has applied the extension principle strictly to the updating step, especially in a hydrogeological setting.

In a Kalman filter, the prior model ($\hat{\mathbf{y}}^-$) is updated ($\hat{\mathbf{y}}$) by adding to the prior model values a weighted difference between the measurements and the prior model values at those measurement locations:

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}^- + \kappa(\mathbf{z} - H\hat{\mathbf{y}}^-) \qquad (1)$$

where κ is the Kalman gain, a matrix that minimizes the covariance of the estimation error, **z** is a vector of measurements, and H is a matrix of ones and zeros that pulls the prior model values at measurement locations out of $\hat{\mathbf{y}}^-$. The algorithm for extending this equation requires individually updating each element of $\hat{\mathbf{y}}^-$ by the measurement values. Thus, there are n implementations of the above equation, one for each value updated. For each element $\hat{y}_i^-$ (i = 1,..,n) of $\hat{\mathbf{y}}^-$, Equation (1) becomes

$$\hat{y}_i = \hat{y}_i^- + \kappa_i(\mathbf{z} - H\hat{\mathbf{y}}^-) \qquad (2)$$

where κ_i represents the ith row of the Kalman gain matrix. The extension principle is applied to Equation (2) in order to produce an a posteriori fuzzy number estimate. Repeating this for every $\hat{y}_i^-$ in $\hat{\mathbf{y}}^-$ provides one with the complete updated fuzzy model.

5. EXAMPLE

Though it does not contradict reality, the example application provided here is comprised of synthetic values. To begin, an appropriate expert provides fuzzy hydraulic conductivity values at sparsely sampled areas. Possible information the expert could use to opine upon the hydraulic conductivity values might pertain to soil type, geologic history, or knowledge of a similar site. Regardless of the evidential motivation behind the expert "measurements," it is assumed that they were provided independent of the crisp measurements at sampling locations. Figure 2 shows a plan view of the site, signifying the locations of fuzzy expert values with triangles and measurement locations with circles. Units of length are given in meters. All samples are taken at the same depth in this two-dimensional example.

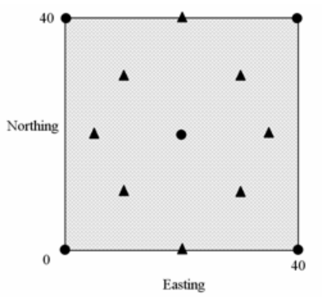

FIGURE 2: Plan view of example problem; triangles are fuzzy value locations, circles are crisp measurement value locations

The fuzzy numbers used to describe the expert's knowledge of the hydraulic conductivity values are all defined by triangular membership functions. As such, the fuzzy numbers can be defined by three parameters: (1) the lower bound of the base of the fuzzy set, (2) the upper bound of the base, and (3) the median value (value with full membership). The locations and said parameters of the fuzzy hydraulic conductivity values are given in Table 1, while the locations and values of the crisp measurements are provided in Table 2.

TABLE 1: Fuzzy number IDs, locations, and parameters

ID	Easting	Northing	Log Lower Bound (m/s)	Log Median (m/s)	Log Upper Bound (m/s)
F1	20	0	-12	-11	-10
F2	10	10	-17	-15	-14
F3	30	10	-9	-7	-6
F4	5	20	-13	-12	-11
F5	35	20	-20	-19	-18
F6	10	30	-10	-8	-7
F7	30	30	-9	-8	-7
F8	20	40	-5	-3	-2

TABLE 2: Crisp measurement IDs, locations, values and errors (variance)

ID	Easting	Northing	Log Measurement (m/s)	Variance (m^2/s^2)
A	0	0	-9	3
B	40	0	-6	4
C	20	20	-25	4
D	0	40	-3	4
E	40	40	-4	3

A sample semivariogram was constructed from the available fuzzy hydraulic conductivity measurements. This semivariogram, with fuzzy semivariance values for different separation distances, h, is shown in Figure 3. The α-cuts plotted for each fuzzy number are $\alpha = \{0, 0.5, 1\}$, the actual plotted values being the upper and lower bounds of each fuzzy set's α-cuts. The model semivariogram fit to this sample is Gaussian with a sill of 30 and a range of 40. It is superimposed upon the model variogram in Figure 3. No nugget is given to the model, because no "measurement" variance is assumed for the expert's input.

With the crisp model variogram defined, the fuzzy values are kriged and a field of fuzzy hydraulic conductivity estimates is generated. This fuzzy model is best represented by a series of contour plots for various α-cuts, as in [*Bardossy et al.*, 1990]. These contour plots, for $\alpha = \{0, 1\}$ are provided in Figure 4, (a) – (c).

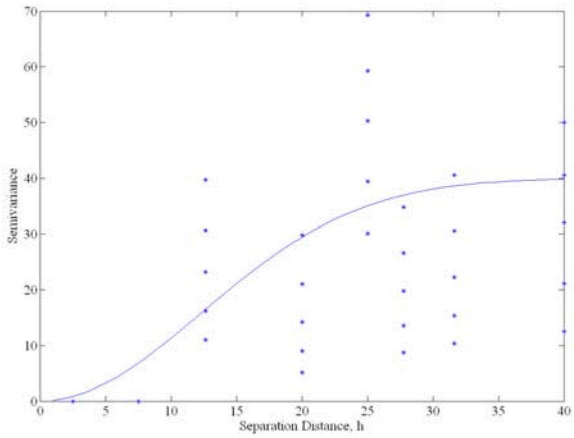

FIGURE 3: Fuzzy sample and crisp model variogram, as prescribed by the fuzzy hydraulic conductivity data.

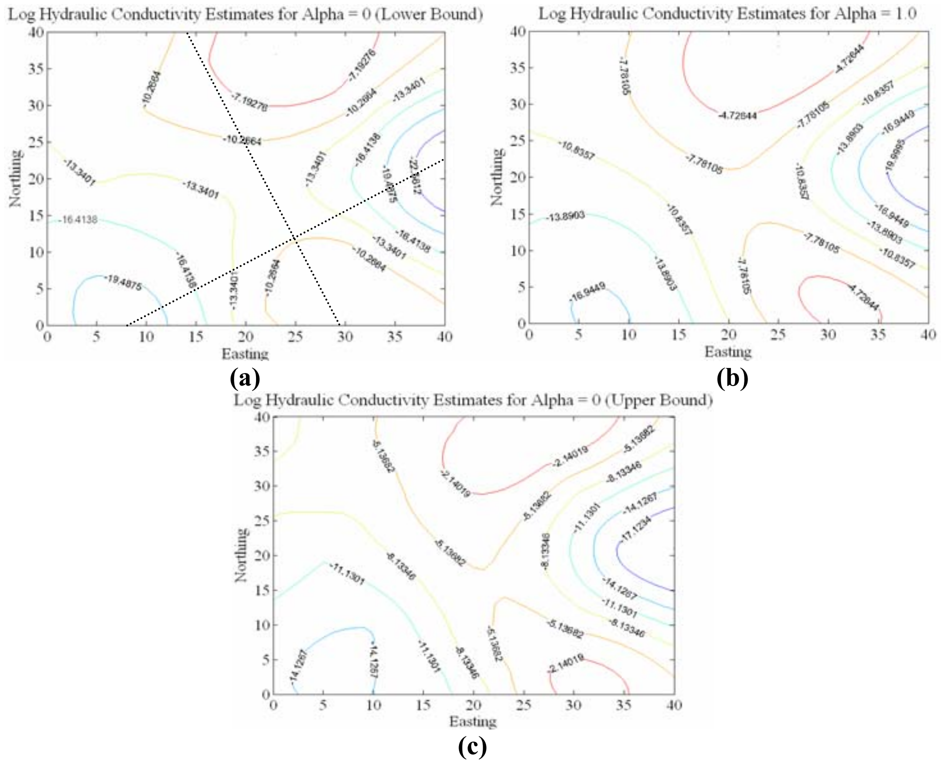

FIGURE 4 (a) – (c): Contour plots of kriged fuzzy estimates of log hydraulic conductivity for (a) lower bound of the zero α-cut, (b) the median values (α = 1), (c) upper bound of the zero α-cut.

Having the fuzzy hydraulic conductivity model in hand, the next step is to incorporate the crisp measurements in order to update the fuzzy model. Applying the updating step of the fuzzy Kalman filter, the fuzzy model is updated by those crisp measurements in Table 2. The resulting contours corresponding to those in Figure 4 are shown in Figure 5 (a) – (c).

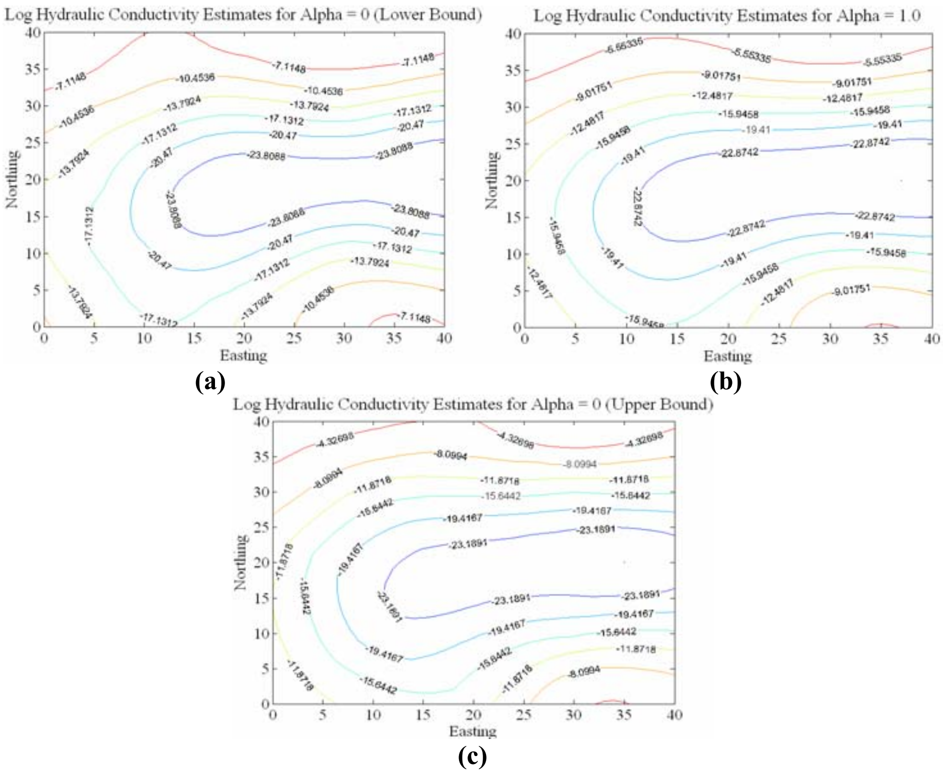

FIGURE 5 (a) – (c): Contour plots of updated fuzzy estimates for (a) lower bound of the zero α-cut, (b) the median values (α = 1), (c) upper bound of the zero α-cut.

6. RESULTS AND CONCLUSIONS

By contrasting the corresponding contours in Figure 4 and Figure 5, two differences are evident. The first difference is intuitive. When crisp measurements are introduced to update the fuzzy estimates, the fuzziness of the estimates (how wide the α-cuts are) can only be reduced since the crisp measurements contain no fuzziness, making the fuzzy model more precise.

Secondly, in the contour plots of Figure 4, there are axes of symmetry, roughly illustrated by the dashed black lines in Figure 4 (a). This symmetry, however, is missing from the corresponding contour plot in Figure 5 (a). This is due to the influence the crisp measurements have upon the final estimates, presumably defining a posterior model that more accurately represents the true trends present at the site.

At measurement locations, the fuzzy estimates are made very precise and close in value to the corresponding crisp measurements. However, unlike the traditional Kalman filter algorithm, these posterior estimates at measurement locations are not made identical in value to the crisp measurements. This is a byproduct of the difference in data types between the measurements and prior estimates, and is not considered an inconsistency. As Figure 6 shows, the posterior fuzzy estimates are quite similar to the corresponding crisp values.

Presented here was an alternative to using fuzzy information in conjunction with crisp data in a hydrogeological setting. By using Kalman filtering, each data type is used in a manner that respects its relative credibility. What results is a rather precise fuzzy field that presumably captures the trends inherent in the model. This fuzzy model may be defuzzified to create a crisp hydraulic conductivity field for input into traditional groundwater flow models, or may be left fuzzy to be used in fuzzy-friendly groundwater flow models [*Dou and Woldt*, 1995; *Dou and Woldt*, 1999].

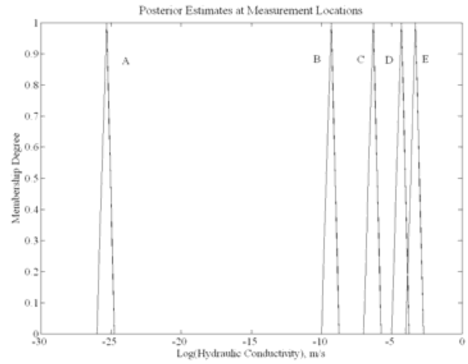

FIGURE 6: Posterior estimates at all measurement locations; Measured values are: -25(A), -9(B), -6(C), -4(D), -3(E)

REFERENCES

Bardossy, A., I. Bogardi, and W. E. Kelly (1989), Geostatistics utilizing imprecise (fuzzy) information, Fuzzy Sets and Systems, 31, 311-328.

Bardossy, A., I. Bogardi, and W. E. Kelly (1990), Kriging with imprecise (fuzzy) variograms II: Application, Mathematical Geology, 22(1), 81-94.

Chen, G., Q. Xie, L. and S. Shieh (1998), Fuzzy kalman filtering, Information Sciences, 109(1-4), 197-209.

Chiang, H. H., S. J. Wu, and T. T. Lee (2003), Application of fuzzy sets theory in process data filtering, Proceedings of the 2003 IEEE International Conference on Fuzzy Systems, 325-330.

Dou, C., W. Woldt, M. Dahab, and I. Bogardi (1995), Steady state groundwater flow simulation with imprecise parameters, Water Resour. Res., 31(11), 2709-2719.

Dou, C., W. Woldt, M. Dahab, and I. Bogardi (1997), Transient ground-water flow simulation using a fuzzy set approach, Ground Water, 35(2), 205-215.

Klir, G., and B. Yuan (1995), *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall of India, New Delhi, India.

Welch, G., and G. Bishop (2001), An introduction to the kalman filter, Course, SIGGRAPH.

Zadeh, L. (1965), Fuzzy sets, Information and Control, 8(3):338-353.