ON DATA ASSIMILATION FOR A 1D-NET RIVER MODEL WITH 2D 'ZOOM' AREAS

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Abstract

We introduce the idea of a 2D 'zoom' into the 1D-net river model and develop basic principles of information exchange. The 'zoom' model is superposed over the main model allowing consideration the 2D effects locally. The main model benefits from using 'zooms', while no modification or change in it is required. Then, we construct the algorithm allowing assimilation of the data measured inside the 'zoom' area, such that both assimilation and coupling problems are solved within the same optimization loop. Results of numerical tests are presented.

1. INTRODUCTION

Although computer power has been steadily growing in recent years, operational hydrological models describing river networks are still based on the 1D shallow water equations (SWE) or the Saint-Venant equations with storage areas. Essentially two-dimensional situations, such as those that occur during flooding, are represented by source terms computed using empirical expressions. This approach may suffer accuracy limitations and, of course, tells us nothing about the 2D flow pattern in the area of interest. Thus, using local 2D SWE models ('zoom' models) coupled in a certain way with the 1D-net global model is justified.

A natural way to introduce 'zoom' models could be the domain decomposition method, when one obtains a set of 1D channels and 2D areas/junctions with or without overlapping [Miglio et al, 2004]. This means, however, that the existing 1D-net model cannot be used as it is and a new mixed model has to be created. We proceed from a practical condition that the 1D-net global operational model must stay intact. We suggest a coupling principle which may be called **'superposition'** rather than **'decomposition'**: we keep the overall unity of the existing 1D model, but source terms to it within a 'zoom' area are estimated via the 2D local solution as a **'defect correction'** [Brandt, 1977]. The 1D model in turn provides a key part of the **characteristic boundary conditions (BC)**, which are required to specify a well-posed 2D problem. Thus, the 2D local model is 'superposed' over the 1D model in a 'zoom' area producing the 2D estimation of the flow and improving performance of the 1D model. An advantage of the proposed method is that while the 1D and the 2D models communicate via available input/output entries, they are actually independent of each other and, therefore, one can use available standard software. Also, the 2D-models could be run in parallel, if necessary. A possible difficulty here is that the two models are not consistent: a) the 1D model cannot provide the full set of BC for the

2D model; b) the 1D model is usually solved on much a coarser mesh with a typical ratio $10^1 - 10^2$ for the space step and $10^2 - 10^3$ for the time step. The lack of information can be compensated using a-priori information or **measured data**.

Thus, first we build the coupling procedure known as the **'wave-form relaxation method (WFR)'** [Bjorhus, 1995] (which is a subset of the global time Schwarz method), when the 1D and the 2D models are solved consecutively in the global time domain each, exchanging information between complete runs (not every time step!). To provide a sufficient set of BC for the 2D model we must use additional assumptions, which are not necessarily valid in general, but may serve quite well as applied to the river hydraulics case. Numerical experiments show an efficiency of the method for the case when no essential inconsistency appears at the model interfaces. At this stage we try to attain the basic principles of information transfer between the 1D and 2D models. Next we turn to the issue of **data assimilation** into a such model, assuming some data is available within the 'zoom' area. The basic idea is to build a **joint assimilation-coupling procedure (JAC)**, which solves simultaneously both data assimilation and coupling problems, rather than the classical data assimilation problem for the already coupled model (obtained using WFR, for example). To do so, we specify an extended objective functional such that in addition to the common data assimilation terms it includes coupling conditions written in a 'weak' integral form. The main advantage of this approach is that additional assumptions are simply not needed, i.e. one can evade the difficulties of coupling inconsistent models! The extended objective functional is minimized using quasi-Newton LBFGS algorithm, while the gradient is computed using the adjoint method. Numerical experiments show that JAC algorithm is equally or less expensive compared to the assimilation procedure for the coupled model. It becomes even more efficient as the inconsistency level grows. Finally, we conduct numerical tests where we consider a toy flooding event that involves overflowing of the main channel and a moving front travelling over previously dry areas. Naturally, this case requires application of a 2D 'zoom' model.

2. PROBLEM STATEMENT

For the problem layout one should consult Fig.1(left). The 2D problem is considered in the domain Ω_2 confined by boundaries Γ_{3-6}, while the main channel (domain Ω_1) is confined by boundaries $\Gamma_{1,2}$. For simplicity we assume that the positions of Γ_3, Γ_4 along the main channel are fixed. The boundaries Γ_5, Γ_6 are time dependent and represent moving wet/dry fronts. The bathymetry is given by function $Z(x, y)$.

The model equations are as follows

$$U_t + A(U)_x + B(U)_y - S(U) = 0, \quad (x, y) \in \Omega_2(t), \quad t \in (0, T) \tag{1}$$

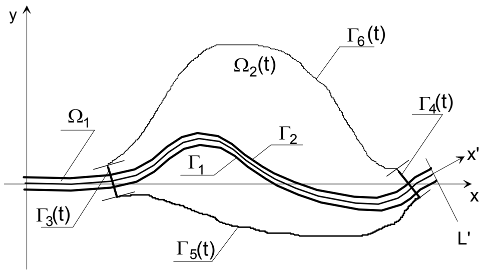
$$U = [\ h,\ q,\ p\ ]^T$$

$$A(U) = \left[\ q,\ q^2/h + gh^2/2,\ qp/h\ \right]^T, \quad B(U) = \left[\ p,\ pq/h,\ p^2/h + gh^2/2\ \right]^T$$

$$S(U) = [\ 0,\ gh(Z_x - f_x),\ gh(Z_y - f_y)\ ]$$

Here h is the surface elevation, q and p are components of discharge, $Z_{x,y}$ and $f_{x,y}$ are the bed slope and the friction slope associated to the x and y axes respectively. Boundary conditions for the 2D SWE problem (assuming that flow at Γ_{3-4} remains subcritical) are:

$$t = 0:\ h(\Omega_2(0), 0),\ q_2(\Omega_2(0), 0),\ p_2(\Omega_2(0), 0)$$



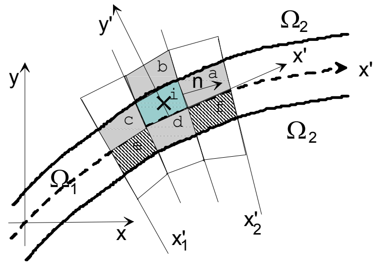


FIGURE 1. Left: problem layout; Right: finite volume mesh

$$\Gamma_3: \quad w_1(\Gamma_3, t) := q_{\vec{n}} + (c - u_{\vec{n}})h, \quad w_3(\Gamma_3, t) := q_{\vec{\tau}}, \; if \; u_{\vec{n}} > 0 \tag{2}$$
$$\Gamma_4: \quad w_2(\Gamma_4, t) := q_{\vec{n}} - (c + u_{\vec{n}})h, \quad w_3(\Gamma_4, t) := q_{\vec{\tau}}, \; if \; u_{\vec{n}} < 0$$

where $w_k(\cdot, t)$ are the 2D SWE characteristic variables. Here $q_{\vec{n}}$ and $q_{\vec{\tau}}$ are the normal and tangent component of the boundary discharge, $u_{\vec{n}} = q_{\vec{n}}/h$ is the normal velocity, $c = (gh)^{1/2}$ is celerity. In the case when the bed and friction slopes are sufficiently mild the set $h(\Omega_2, 0)$, $q(\Omega_2, 0)$, $p(\Omega_2, 0)$, $w_1(\Gamma_3, t)$, $w_3(\Gamma_3, t)$, $w_2(\Gamma_4, t)$, $w_3(\Gamma_4, t)$ should specify a well-posed 2D SWE problem.

The 1D model can be derived from the 2D SWE model in two steps. First, the 2D model has to be considered in the channel-following coordinate system (x', y'). Assuming that the median curve of the channel is given in the parametric form $x_m = m_1(x'), \; y_m = m_2(x')$ we obtain the following transformation

$$\frac{\partial x'}{\partial x} = \frac{cos(\alpha')}{1 - y'\alpha'_x}, \quad \frac{\partial y'}{\partial x} = -sin(\alpha'), \quad \frac{\partial x'}{\partial y} = \frac{sin(\alpha')}{1 - y'\alpha'_x}, \quad \frac{\partial y'}{\partial x} = cos(\alpha')$$

where α' is the angle between the x-axis and the local tangent to the median curve, $\alpha'_x = \partial\alpha'/\partial x$. By neglecting $y'\alpha'_x$ (that is often justified for river flows), we obtain the same equations as (1) for the new variables $U' = [\; h, q', p' \;]^T$. The second step is to integrate these equations on y' from Γ_1 to Γ_2. Assuming a) zero fluxes through $\Gamma_{1,2}$; b) $u'_{y'} = 0$; c) $(h_{x'})_{y'} = 0$ we get the Saint-Venant equations or, in the case when the main channel has a constant rectangular cross-section wide b, the 1D SWE as follows

$$\tilde{U}'_t + \tilde{A}(\tilde{U}')_{x'} - \tilde{S}(\tilde{U}') = \Psi, \quad (x') \in \Omega_1, \quad t \in (0, T) \tag{3}$$
$$\tilde{U}' = [\; H', \; Q' \;]^T, \quad \Psi = [\psi_1, \; \psi_2]$$
$$\tilde{A}(\tilde{U}') = \left[\; Q', \; (Q')^2/H' + g(H')^2/2 \;\right]^T, \quad \tilde{S}(\tilde{U}') = [\; 0, \; gh(Z'_x - f'_x) \;]$$

Here $\Psi = [\psi_1, \; \psi_2]$ are the source terms introduced to control the 1D solution. We assume that these are existing entries into the standard 1D model. Boundary conditions for the 1D SWE problem are:

$$t = 0: \; H'(x', 0), \; Q'(x', 0)$$
$$x' = 0: \; W'_1(0, t) := Q' + (c' - u')H' \tag{4}$$
$$x' = L': \; W'_2(L', t) := Q' - (c' + u')H' \tag{5}$$

where $u' = Q'/H'$ and $c' = (gH')^{1/2}$.

3. INFORMATION EXCHANGE PRINCIPLES

a) 2D → 1D information transfer

To explain the information exchange between the 1D and 2D SWE models we turn directly to the finite-dimensional representation of problems given in (1), (3). Let us consider a finite volume mesh covering the local 2D area (Ω_2) in such a way that volume interfaces constitute boundaries of the main channel $\Gamma_{1,2}$ as shown in Fig.1(right). For the i^{th} finite volume the model equations (1) are approximated as follows

$$U_i(t+dt_2) = U_i + dt_2\, G(U_i); \quad G(U_i) := (\sum_{k=1}^{4} F_k(U_i)\, l_k + S(U_i)) \tag{6}$$

where dt_2 is the time step used for the 2D model, $G(\cdot)$ is a 2D SWE discrete space operator, $F_k(U_i)$ are U-fluxes via k^{th} edge of the i^{th} volume, l_k is the length of the edge. For the edge we define a rotation $T(\alpha)$, where α is the angle between the normal to the edge and the x-axis. Variables $V = [\, h,\ q_{\vec{n}},\ q_{\vec{\tau}}]$ can be used to define a vector of local Godunov fluxes as follows

$$\Phi(V) = \left[q_{\vec{n}},\ q_{\vec{n}}^2/h + gh^2/2,\ q_{\vec{n}}q_{\vec{\tau}}/h\right]^T$$

Computing of $F(U_i)$ consists of three steps [Toro, 2001]. First we compute the normal and tangent discharge components in two volumes adjacent to the edge as follows $V = T(\alpha)U_i$. Then we compute $\Phi(V)$ as the exact solution of the local Riemann problem. The last step is to compute U-fluxes using rotation $T^{-1}(\alpha)$, so we can eventually write

$$F_k(U_i) = T^{-1}(\alpha_k)\Phi(T(\alpha_k)U_i) \tag{7}$$

For $\forall i \in \Omega_1$ the variables in the channel-following coordinates U_i', U'-fluxes $F_k(U_i')$ and $G(U_i')$ can be obtained using the rotation $T(\alpha')$ as follows

$$U_i' = T(\alpha')U_i, \quad F_k(U_i') = T(\alpha')F_k(U_i), \quad G(U_i') := \sum_{k=1}^{4} F_k(U_i')l_k + S(U_i')$$

The 1D model can be represented in the finite-dimensional form as follows

$$\tilde{U}_j'(t+dt_1) = \tilde{U}_j' + dt_1\, (\tilde{G}(\tilde{U}_j') + \Psi_j); \quad \tilde{G}(\tilde{U}_j') := \sum_{k=1}^{2} \tilde{F}_k(\tilde{U}_j')\, b + \tilde{S}'(U_j') \tag{8}$$

where dt_1 is the time step used for the 1D model, $\tilde{G}(\cdot)$ is a 1D SWE discrete space operator, $\tilde{F}_k(\tilde{U}_j')$ are the $\tilde{U}'$-fluxes via k^{th} edge of the j^{th} 1D volume, which are defined by (7) assuming that $\alpha_1 = 0$, $\alpha_2 = \pi$.

We compute sources Ψ_j to the 1D model (8) as a 'defect correction' (as it is defined in the multigrid method) considering the 2D 'zoom' model as a finer model. Thus, we define a projection (restriction) operator that maps U' and $G(U')$ into the 1D space by averaging these quantities over the j^{th} volume

$$I_j(\cdot) = \frac{1}{dt_1}\frac{1}{dx_1}\frac{1}{b}\int_{t_j}^{t_{j+1}}\int_{x_1'}^{x_2'}\int_{-b/2}^{b/2}(\cdot)dy'dx'dt \tag{9}$$

The 1D variables are (H', Q'), hence we retain only two first components of $I_j\, U'$ and $I_j\, G(U')$. Next we compute the action of the 1D (coarse) operator $\tilde{G}(I_j\; U')$ and define finally the defect correction as follows

$$\Psi_j = I_j\, G(U') - \tilde{G}(I_j\, U') \tag{10}$$

Let us note that for consistent grids the main part of Ψ represents the overflow fluxes via the boundaries $\Gamma_{1,2}$, since these fluxes are accounted by $G(\cdot)$, but are not defined in $\tilde{G}(\cdot)$.

b) 1D → 2D information transfer

For the boundaries Γ_3 - 'inlet' and Γ_4 - 'outlet' one can write

$$\int_{-b/2}^{b/2} w_k(x', y', t)\, dy' = bW_k(x', t)|_{x' \in \Gamma_{k+2}},\; k = 1, 2 \tag{11}$$

where $W_{1,2}(x', t)$ are defined as in (3)-(4), but for arbitrary x'. Let us note that no more information can be extracted from the 1D model. The distribution of $w_{1,2}$ on y' remains unknown as well as w_3 in (3), since there exists no related quantity in the 1D formulation.

4. WFR (SCHWARZ) COUPLING METHOD

We will consider the initial condition being known both for 1D and 2D models and assume that boundary conditions at the ends of the 1D section $W_1(0, t)$, $W_2(L', t)$ are known. Here we solve a classical coupling problem. We need to guess on the distribution $w_{1,2}$ on y' (we know the integral values in (11)!) and specify also w_3. For example, one can assume that $w_{1,2}$ are distributed on y' uniformly (or $\propto h^{3/2}$) and $w_3|_{\Gamma_{3,4}} \equiv 0$. Apparently, these assumptions could be justified only if the physical conditions of the flow are quite similar. Here we face a basic difficulty of coupling inconsistent models. After we have decided on the 2D model BC, the WFR method consists of the steps:

a) given the initial condition $\tilde{U}'(x', 0)$, $x' \in (0, L')$, boundary conditions $W_1(0, t)$, $W_2(L', t)$ solve the 1D problem (3) for $t \in (0, T)$ assuming that $\Psi(x', t) = 0$, $x' \in \Omega_2$;

b) compute boundary conditions for the 'zoom' model $w_1(\Gamma_3, t)$, $w_2(\Gamma_4, t)$ using constraints (11) complemented with a certain distribution rule (uniform distribution, for example), a-priori specify $w_3(\Gamma_3, t)$, $w_3(\Gamma_4, t)$;

c) given the initial condition $U(\Omega_2, 0)$, boundary conditions $w_1(\Gamma_3, t)$, $w_2(\Gamma_4, t)$, $w_3(\Gamma_3, t)$, $w_3(\Gamma_4, t)$ solve the 'zoom' problem for $t \in (0, T)$;

d) compute source terms to the 1D model $\Psi(x', t)$, $x' \in \Omega_2$ for the current 2D 'zoom' solution using (10);

e) return to a) and repeat iterating until the solution of both models stops changing significantly.

5. JOINT ASSIMILATION-COUPLING PROCEDURE (JAC)

Assimilation of data into the 1D-net model with 'storage areas' could become complicated due to the following reasons: a) 'storage areas' described as sources terms may be non-differentiable; b) it could be difficult to relate the local measured quantities (velocities, local elevation) to the 1D variables (H', Q') particularly in those areas, where the 1D approximation of a flow is not quite adequate.

We suggest an approach that allows assimilating local data measured within the 2D 'zoom' areas into the 1D-net model. Let us suppose that the BC at the ends of the 1D segment which includes a 'zoom' area are sought. Since the 1D model cannot provide sufficient information to specify uniquely BC of the 2D 'zoom' model, we consider them as additional unknown controls considering (11) as weak constraints. On the other hand, the 2D 'zoom' model contains superfluous information (in respect to the 1D model), thus we retain (10) as a strong constraint. We introduce a generalized objective functional

$$J = \gamma J^* + J_1 + J_2 \tag{12}$$

This functional comprises a regular data assimilation term (weighted by γ)

$$J^* = \sum_m \int_0^T \beta_m (U_m - \hat{U}_m)^2 dt \tag{13}$$

where m is the number of a finite volume where measurements $\beta_m \hat{U}_m$ are available, β_m are weights, and coupling conditions (11) in a 'weak' from as follows

$$J_k = \int_0^T \Big(\int_{-b/2}^{b/2} w_k(x', y', t)\, dy' - bW_k(x', t)|_{x' \in \Gamma_{k+2}} \Big)^2 dt, \quad k = 1, 2 \tag{14}$$

This arrangement leads us to a 'one-way relaxed' model formulation, which can be defined by the following steps:

a) given the initial condition $U(\Omega_2, 0)$ and a current estimation of boundary conditions $w_1(\Gamma_3, t)$, $w_2(\Gamma_4, t)$, $w_3(\Gamma_3, t)$, $w_3(\Gamma_4, t)$ solve the 2D 'zoom' problem for $t \in (0, T)$, keeping the 2D flow field values at the sensor locations;

b) compute source terms to the 1D model $\Psi(x', t),\ x' \in \Omega_2$ for the current 2D 'zoom' solution using (10);

c)given the initial condition $\tilde{U}'(x', 0),\ x' \in (0, L')$, current estimation of boundary conditions $W_1(0, t)$, $W_2(L', t)$ and the correction $\Psi(x', t)$ solve the 1D problem (3) for $t \in (0, T)$;

d) compute the value the generalized objective functional.

The code implementing the model a)-d) must be differentiated in order to produce the adjoint code, which computes gradients of J in respect with all control entries. It must be underlined that no additional assumptions have been used to define this model.

6. NUMERICAL EXAMPLES

We use a finite volume solver based on the weighted average flux (WAF) method described in [Toro, 2001] that is modified to include bathymetry treatment. Friction term is approximated using a semi-implicit scheme. In order to simplify implementation numerical experiments are conducted for a uniform rectangular mesh. For all tests we use the bathymetry shown in Fig.2(left). The boundary at $y \leq 0(\Gamma_1)$ is a wall. To enable analysis of results we compute a 'reference' flow by solving the 2D problem in the entire domain $\Omega_1 \cup \Omega_2$. As the initial condition we use a steady-state flow confined by the main channel. This flow is supported by a constant value of the inlet boundary control (at the outlet we always assume an 'open boundary'). Then we add a time-dependent component, which creates a wave propagating downstream. When the wave reaches the 'low bank' it starts overflowing and produces a wet/dry front travelling over the previously dry area. The discretization steps used for the reference solution are $dx = 20m$, $dy = 40m$, $dt = 0.1s$. A flow pattern at $t = 600s$ is shown in Fig.2(right).

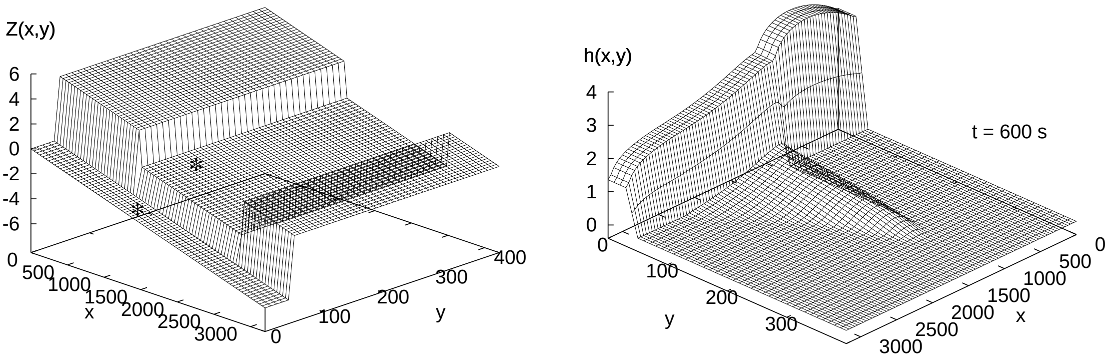


FIGURE 2. Bathymetry(left) and reference flow (right)

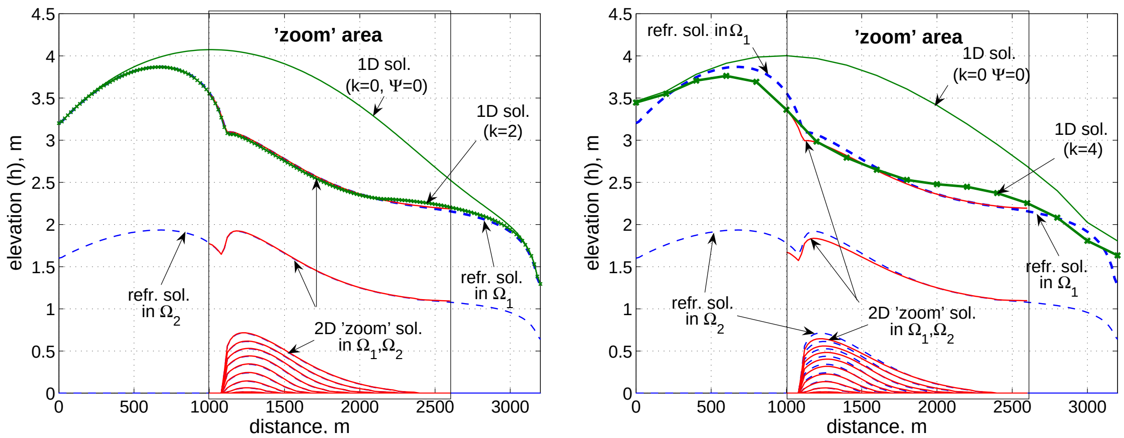


FIGURE 3. Reference flow and the 1D/2D 'zoom' solutions by WFR method for consistent (left) and inconsistent mesh (right)

a) Given the BC of the 1D model, we have tested the WFR algorithm (Section4). First, we computed the 1D and 2D 'zoom' solutions using the same mesh as in the reference case. The results are presented in Fig.3(left). Here we show: the reference solution (in dashed line), the initial approximation of the 1D solution ($k = 0, \Psi = 0$), the 1D solution after two iterations of WFR method ($k = 2$) and the corresponding 2D 'zoom' solution. One can see that the 1D solution follows closely the reference solution within the main channel area. The 2D 'zoom' solution is very close to the reference solution within the 'zoom' area. Numerous tests show that this method actually converges in few iterations approaching the theoretical limit for WFR (two iterations). Next, we applied the WFR algorithm when models were solved on different meshes. While the mesh of the 2D 'zoom' model corresponds to the reference case, the 1D mesh is much coarser with the following ratios $dx_1/dx_2 = 10$, $dt_1/dt_2 = 100$. The results are presented in Fig.3(right). For this case the convergence rate is a bit slower (4 iterations were needed) and the difference between solutions can be clearly seen.

b) In order to verify performance of the JAC algorithm we conducted identical twin experiments. Now we assume that BC of the 1D model are not known, but data is available

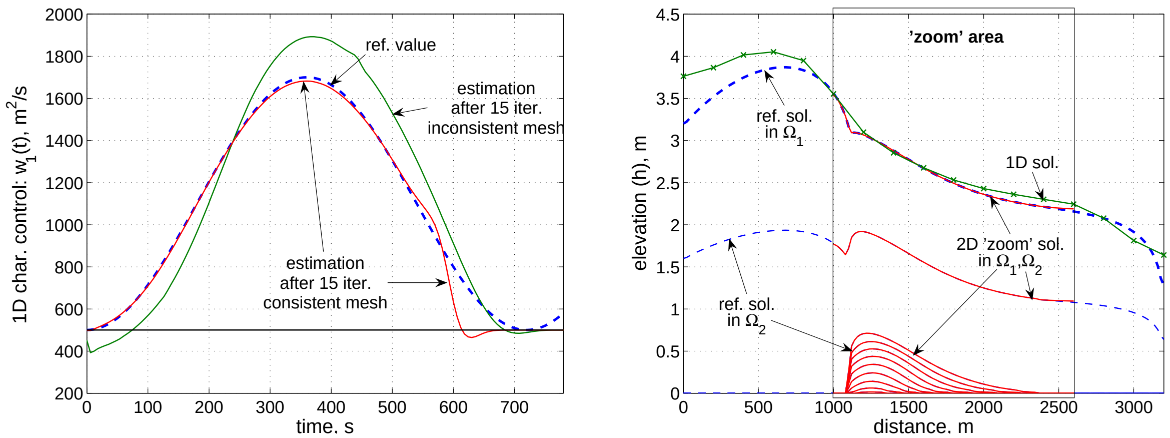


FIGURE 4. Reference boundary condition and its estimations for consistent/inconsistent mesh (left); reference flow and the 1D/2D 'zoom' solutions by JAC for inconsistent mesh (right)

within the 'zoom' area from two sensors located as shown in Fig.2(left)(by '*' label). In the present case we assume that all state variables are measured. For minimization (12) we use the LBFGS routine [Gilbert and Lemaréchal, 1989] guided by adjoint sensitivities. The adjoint code has been obtained by means of automatic differentiation (TAPENADE) applied directly to the model **a)-d)** as described in Section 5. The results of DA are presented in Fig.4. To the left, one can see the reference BC (in bold dashed line) and its retrieved values by 15 iterations of the JAC algorithm. Up to this point the initial value of J had been decreased by factor 10^4 and, although the process continues to converge, results remain visually the same. We consider two cases: consistent and inconsistent meshes. To the right, the 1D and the 2D 'zoom' solutions related to the retrieved value for the latest case are presented, as well as the reference solution at $t = 600s$. We note that the 'zoom' solution cannot be distinguished from the reference value, while the 1D solution is much closer to it in the main channel area as compared to Fig.3(right). Obviously, the 'zoom' solution is dominated by measured data. It 'attracts' the 1D solution within the 'zoom' area. The quality of the BC estimation, however, depends on the 1D mesh. Since the WFR coupling method claims 3-4 inner iterations applied both to the forward and adjoint models, the overall computational cost of the JAC algorithm must be similar or less as compared to a DA applied to a fully coupled model.

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