HYPERCOMPLEX APPROXIMATION TO VORTICITY

LEONARDO TRAVERSONI[1]

[1]Univedrsidad Autonoma Medtropolitana (I)

Abstract

We present a new approach to vortex methods using quaternionic analysis in order to represent Navier Stokes equations only depending on vorticity as a quaternionic variable. There are great computational advantages on doing that as well as more accurate approximation. This approach also allows a more natural use of several computational tools as wavelets, particularly what is called quaternionic multiresollution analysis. Several computational results are shown.

1. CLASSICAL QUATERNION VIEW OF NAVIER STOKES

There is a "classic" approach originally developed by [Guerlebeck 1998] and Sproessig as follows:

1.1. **Stokes.** Let $G \subset R^n$ be a bounded domain $u = (u_1, ..., u_n)$ a vectorial function and p a scalar function. In R^3, u is the velocity of the fluid and p is the hydrostatic pressure. Let f be the external forces function and η the fluid's viscosity. The Stokes boundary values problem is set as:

$$-\Delta u + \frac{1}{\eta} grad\, p = f \;\; en \;\; G$$
$$div\, u = 0 \;\; en\, G$$
$$u = 0 \;\; en\, \Gamma$$

Its hypercomplex representation is:

$$DDu + \frac{1}{\eta} Dp = f \;\; en\, G$$
$$Sc\, Du = 0 \;\; en\, G$$
$$u = 0 \;\; en\, \Gamma$$

Here D is the differential and means to do a left multiplication by the quaternion $\{0, D_x, D_y, D_z\}$ as a result of the non conmutativity of the quaternionic multiplication we can factorize the Laplacian as DD

we assume that $f = \sum_{i=1}^{n} f_i e_i$

we have that if u is a solution of this equations then:

$$u = T_G \mathbf{Q} T_G f - \frac{1}{\eta} T_G \mathbf{Q} p$$

we can write the second Stokes equation as:

$$-Sc\,(\mathbf{Q}T_G f - \frac{1}{\eta}T_G\mathbf{Q}p = 0$$

Theorem

The Stokes system has a solution, it is unique be it $\{u, p\}$ where p is unique up to a constant and may be represented by the equation:

$$u = T_G\mathbf{Q}T_G f - \frac{1}{\eta}T_G\mathbf{Q}p$$

and is true that:

$$\left(\frac{\lambda_1}{\lambda_1 + 1}\right)^{1/2} ||u||_{2,1} + \frac{1}{\eta}||\mathbf{Q}_p||_2 \leq \sqrt{2}||T_G f||_2$$

where λ_1 is the first eigenvalue of the Dirichlet problem for the Laplacian.

1.2. **Navier Stokes equations.** Now we have to solve the boundary value problem using hypercomplex integral transformations

All the following is also valid for the non linear case

We choose the harmonic time dependent case of the problem of a flux with free convection

The idea now is to transform the sistem of differential equation in an equivalent system of integral operators

The problem:

$$-\Delta\underline{u} + a_1(\underline{u}\cdot grad)\underline{u} + f(u) + a_2\,grad\,p + a_3(-e_3)w = F(x) \;\; in\,G$$

$$div\,\underline{u} = 0 \quad in\,G$$

$$-\Delta w + a_4(\underline{u}\cdot grad)w = g \;\; in\,G$$

$$\underline{u} = 0 \;\; on\,\Gamma$$

$$w = 0 \quad on\,\Gamma$$

The coefficients a_i correspond to physical constants :

Be ρ the fluid density, η the viscosity, γ the Grashof number, k the thermic conductivity and m the Prandtl number. We have then:

$$a_1 = \frac{\rho}{\eta}\,,\; a_2 = \frac{1}{\eta}\,,\; a_3 = \frac{\gamma}{\eta}\,,\; a_4 = \frac{m}{k}$$

p is the hydrostatic pressure and $\underline{u}$ is the fluid velocity and w the temperature.

Now we change to Clifford Algebras notation:

We first do:

$$u = u_0 + \underline{u}$$

adding to the system:

$$\Delta u_0 = 0 \;\; en\, G$$
$$u = 0 \;\; en\, \Gamma$$

p is now the quaternionic function:

$$(p, 0, 0, 0)^T$$

defining:

$$M(u) = a_1(\underline{u} \cdot grad)u + f(u) - F$$

we can write:

$$D^2 u + M(u) + a_2\, grad\, p + a_3(-e_3)w = 0 \;\; in\, G$$
$$-ScDu = 0 \;\; in\, G$$
$$D^2 w - a_4 Sc(uD)w = g \;\; in\, G$$
$$u = 0 \;\; on\, \Gamma$$
$$w = 0 \;\; on\, \Gamma$$

we now obtain a non linear system of hypercomplex integral operators:

$$u = -TQT[M(u) - a_3 e_3 w] - a_2 TQp$$
$$0 = Sc\{QT[M(u) - a_3 e_3 w] + a_2 Qp\}$$
$$w = -a_4 TQTSc(uD)w + TQTg$$

the border conditions are fullfilled because of the orthoprojection properties of Q:

$$Qu = Dv$$
$$tr_\Gamma TQu = tr_\Gamma TDv = tr_\Gamma(w - F_\Gamma tr_\Gamma w) = 0$$

Depending on the particular values of a_i we can describe other problems:

If we have $a_3 = 0$ the first 2 equations describe the stationary problem of Navier Stokes If also $a_1 = 0$ then it is Stokes problem. If all the a_i are zero then it is the Poisson problem.

1.3. **Iterative Method.** The procedure is based on the Banach fixed point. Starting from the previously chosen functionsu_0 and w_0 we calculate for $n = 1, 2, ...$

$$u_n = -TQT[M(u_{n-1}) - a_3 e_3 w_{n-1}] - a_2 TQp_n$$
$$0 = Sc\{QT[M(u_{n-1}) - a_3 e_3 w_{n-1}] + a_2 Qp_n\}$$
$$w_n = -a_4 TQTSc(u_n D)w_n + TQTg$$

to calculate w_n we use a nested iteration:

$$w_n^{(i)} = -a_4 TQTSc(u_n D)w_n^{(i-1)} + TQTg \;\; (i = 1, 2, ...)$$

In each step of the iteration we have to solve a Stokes problem and a Poisson equation.

2. VORTICITY

Vortex methods have been since the 70's an alternative in certain cases to finite elements or finite differences, as quaternions are well suited to represent vorticity we will use them. In order to do that we consider now the non compressible flow equations:

$$\frac{Du}{Dt} = -\bigtriangledown P + \frac{1}{R}\bigtriangledown^2 u \;\; in\, D$$
$$\bigtriangledown \cdot u = 0 \;\; in\, D$$
$$u = 0 \;\; on\, \partial D$$

Taking ξ as:

$$\xi = \bigtriangledown \times v$$

the vorticity we obtain the vorticity transport equation:

$$\frac{D\xi}{Dt} = (\xi \cdot \bigtriangledown)u + \frac{1}{R}\bigtriangledown^2 \xi$$

Here u is the velocity, P the pressure, R the Reynolds number.
As $\bigtriangledown \cdot u = 0$ and $\xi = \bigtriangledown \times u$
exists a vectorial function $\psi(x)$ such that $u = \bigtriangledown \times \psi$ then

$$\bigtriangledown^2 \psi = -\xi$$

In 3D ψ is the potencial of velocity potencial in 2D is the stream function. The solution is:

$$\psi(x,t) = \int L(x-z)\xi(z)dz$$

where:

$$\left(\frac{-1}{2\pi} log|x| x \in R^2 \frac{1}{4\pi}\frac{1}{|x|} x \in R^3 \right)$$

because

$$u = \bigtriangledown \times \psi$$

we have that:

$$u(x,t) = \int K(x-z)\xi(z)dz$$

where K is:

$$K(x) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|x|^2} \quad x \in R^2$$
$$K(x) = \frac{1}{4\pi|x|^3}\left(0x_3 - x_2 - x_3 0 x_1 x_2 - x_1 0 \;\; x \in R^3\right.$$

The kernel K is singular for both dimensions. This idea leads to the vorticity equations. The transformation leads to a non linear system of differential equations for both variables u and ξ

3. QUATERNION REPRESENTATION OF VORTICITY

Applying ideas of quaternionic and Clifford analysis we search a transformation into one non-linear equation depending only the vorticity ξ. To achieve such goal we use the higher-dimensional version of the Borel-Pompeiu formula:

$$TDu(x) = u(x) - Fu(x) \tag{1}$$

where T is the T-operator (Teodorescu transform), D the Dirac-operator and F the Cauchy integral. The Cauchy integral depends only on the boundary values of u.

That means that if $u = 0$ on the boundary then this part can be deleted of the formula.

Moreover, Du means for a quaternion valued function $(0, u)$ (u is the vector of velocity)

$$Du = (-div\, u,\ rot\, u) \tag{2}$$

As we are working with divergence free vectors and consequently

$$Du = (0,\ rotu)$$

Remembering that

$$rot\ u = \nabla \times u$$

we have that

$u = TDu$ and with $Du = rot\, u = \xi$ it follows

$$u = T\xi \tag{3}$$

This is an expression to describe the velocity u explicitly by the vorticity ξ. If the boundary values of u are not zero but some known quantity then we have this additional known summand F(boundary values of u). The operators T and F are defined as:

$$(T_G u)(x) = -\int_G e(x-y)u(y)dG_y$$

$$(F_\gamma u)(x) = \int_\gamma e(x-y)\alpha(y)u(y)d\gamma_y$$

α is the outer normal to γ at the point y and $e(x)$ the fundamental solution (generalized Cauchy kernel) of the Dirac-operator.

In this way substituting in the above equations we obtain a nonlinear equation in ξ instead a system in u and ξ. To find representation formulas and numerical methods for ξ is one of the goals of the project. Because we have to evaluate only the vorticity (and not in addition the velocity, too) a better efficiency of this approach is expected. Now having this equations we are going to approximate its solution with :

4. QUATERNION MULTIRESOLLUTION ANALYSIS

Starting from my preliminary ideas [Traversoni 1994] about quaternion wavelets and the more general formulation by [Mitrea 1994] a theory about quaternion wavelets may be introduced as follows.

We will consider $H = L^2(R^m)$ with the involution given by the conjugation of complex valued functions and:

$$B(f,g) = \int_{R^m} f(x)b(x)g(x)dx \quad f,g \in L^2(R^m)_{(2)}$$

here $b : R^m \to R^{2+1} \subset C_{(2)}$ is a L^∞ function with $Re\ b(x) \geq \delta > 0$. B is a δ-accretive form on $L^2(R^m)_{(2)}$. Consider now $\{V'_k\}_k$ a multiresolution analysis of $L^2(R^m)$ that is a family of closed subspaces of it for which:

1)$\bigcap_{-\infty}^{+\infty} V'_k = \{0\}$ and $\bigcup_{-\infty}^{+\infty} V'_k$ is dense in $L^2(R^m)$

2)For any $k \in Z$, $f(x) \in V'_k \iff f(2x) \in V'_{k+1}$

3)For any $j \in Z$, $f(x) \in V'_k \Rightarrow f(x-j) \in V'_k$

4)There exists $\phi(x) \in V'_0$ such that $\{\phi(x-j)\}$ is an orthonormal basis for V'_0

The functions :

$$\phi_{j,k} = 2^{km/2}\phi(2^k x - j)\ , \quad k \in Z\ ,\ j \in Z^m$$

form an orthonormal basis for V'_k and there exist $2^m - 1$ functions$\{\psi_\epsilon\}$ in V'_1 having the same type of regularity and decay of ϕ which form an orthonormal basis of the wavelet space $W'_0 = V'_1 \ominus V'_0$

Note that $\{\psi_{\epsilon,j,k}\}_{\epsilon,j}$ with:

$$\psi_{\epsilon,j,k}(x) = 2^{km/2}\psi_\epsilon(2^k x - j)\ , \quad k \in Z\ , \quad j \in Z^m$$

is an orthonormal basis for $W'_k = V'_{k+1} \ominus V'_k$

It is important to note that for the above QMRA of $L^2(R^m)_{(2)}$ it can be prooved that there exists a dual pair of wavelet bases $\{O^L_{\epsilon,j,k}\}_{\epsilon,j,k}$ and $\{O^R_{\epsilon,j,k}\}_{\epsilon,j,k}$ which are r-regular.

Now we will apply the above to our problem, that is we are going to use this multiresolution analysis to approximate the solution of the vortical equations described in the previous section that depend only on the vorticity. The key is then to chose the best weighting function, in this case we think (following the ideas of [Chui 1992])they are b-splines of the type:

A traslational spline given by:

$$Q^{(i)}_{tras} = 2 + \epsilon \sum_{j=0}^{3} d^3_j \left(\frac{t - t_i}{t_{i+1} - t_i}\right) \vec{p^{(i)}_j}$$

and a rotational given by:

$$Q^{(i)}_{rot} = 2 + \epsilon \sum_{j=0}^{3} d^3_j \left(\frac{t - t_i}{t_{i+1} - t_i}\right) C^{(i)}_j$$

to have the particular expression of this b-splines the coefficients $\vec{p(i)}_j$ and $C^{(i)}$ must be found solving two systems of linear equations using the field data.

5. FILTERING MOVEMENTS

The use of quaternion wavelets have several advantages, for example the possibility of filtering components of a complicated movement, in our case for example tides, wind waves or similar known movements of the water that are common to a particular zone and that we know will be present as a component in the observed and more complicated movement. Let then formulate the weighting function as:

$$\phi(x) = \sqrt{2} \sum_{n \in z} h_n \phi(2x - n)$$
$$\psi(x) = \sqrt{2} \sum_{n \in z} g_n \phi(2x - n)$$

Where h and g are quaternion valued constants called low and high pass filter respectively.

This filters may be determined solving some systems of equations for each coordinate of the quaternion starting from the form:

$$H(\xi) = \frac{1}{\sqrt{2}} \sum_{n=0}^{2N+1} h'_n e^{-1n\xi} = \frac{1}{\sqrt{2}} \left(\sum_{n=0}^{N} h_n e^{-1n\xi} + \sum_{k=N+1}^{2N+1} h_{2N+1} - k e^{-1k\xi} \right)$$

6. CONCLUSIONS

The future work will be to calibrate the models and do several trials with different weighting functions.

It is important to note that what we are solving is a function to describe the field of vorticity to obtain the velocity or the pressure we have to do the process to the inverse in order to describe them.

References

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