A BGK MODEL FOR SHALLOW WATER FLOWS: TWO-DIMENSIONAL VERTICAL PLANE PROBLEMS

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ABSTRACT

This paper focuses on illustrating how the Boltzmann theory can be used to formulate numerical algorithms for two-dimensional shallow water equations in the vertical plane. The paper begins by showing that the shallow water equations in the vertical plane are obtainable from the moments of the Boltzmann Bhatnagar-Gross-Krook (BGK) equation. This connection is then exploited to formulate a finite volume model for shallow flows in which the fluxes are obtained on the basis of the BGK equation. The resulting scheme is illustrated using a variety of surface water problems. The advantages of using the Boltzmann-based models to formulate numerical algorithms for surface water flows are summarized.

Keywords: Boltzmann theory; shallow water equations; σ-coordinate.

1. INTRODUCTION

Existing numerical models for hydraulic applications are mostly based on the direct discretization of the macroscopic governing equations such as the shallow water equations, and the Navier-Stokes equations. Boltzmann methods provide an alternative approach to the formulation of hydraulic models. The Boltzmann-based approach (e.g. Chen and Doolen 1998, Ghidaoui et al. 2001, and Zhou 2004) entails the following steps: (a) write a Boltzmann-like equation for the problem under study, (b) ensure that this Boltzmann-like equation is consistent with the known macroscopic equations of the problem being studied, (c) discretize the Boltzmann-like equation, and (d) take appropriate moments to arrive at the desired numerical algorithms. Although this approach may appear cumbersome and unnecessarily indirect at first glance, it has notable advantages such as: (a) the fluxes obtained from the Boltzmann models can resolve both wave and diffusive effects, (b) there is no need for characteristic decomposition, (c) the formulation entails solving a single scalar equation rather than a system of nonlinear equations, and (d) the formulation of Boltzmann-based numerical algorithms in irregular grids is relatively straightforward. It is important to note that the Boltzmann approach is not limited to the formulation of numerical schemes. For example, Boltzmann methods have been used to model the physics of sediment transports (Ni et al. 2000) and porous media flows (Kang 2004). Recently, Chen et al. (2004) showed that the renowned k-ε turbulence model can also be recovered from the Boltzmann equation if one exploits the analogy between molecular fluctuations and turbulent fluctuations.

This paper shows that the shallow water equations in the vertical plane can be obtained from the moments of the Boltzmann equation. This connection is exploited to formulate a numerical model for shallow flows on the basis of the Boltzmann equation. The main idea is that since the continuum equations are moments of the Boltzmann equation, then it is plausible to determine the discrete form of the continuum equations by taking the moments of the discrete Boltzmann equation. The proposed numerical model is applied to linear waves, laminar flows, turbulent flows and seiche.

2. CONNECTION BETWEEN BOLTZMANN EQUATION AND SHALLOW WATER EQUATIONS

The traditional BGK-Boltzmann equation (Vincenti and Kruger 1965) is transformed into σ-coordinates,

$$\frac{\partial hf}{\partial t^*} + \frac{\partial c_x hf}{\partial x^*} + \frac{\partial c_\sigma hf}{\partial \sigma} = \frac{q-f}{\tau} h \qquad (1)$$

where h is the water depth, f is the non-equilibrium particle distribution function, q is the equilibrium particle distribution function, τ is the particle collision time. The σ-coordinate (x^*, σ^*, t^*) is defined as $x^* = x$, $t^* = t$ and $\sigma = (z - \eta)/h$ where z is vertical coordinate and $\eta = B + h$ is the free surface displacement with B the bottom elevation. The superscript asterisks are dropped hereafter for convenience. (c_x, c_σ) is particle velocity in σ-coordinate and the particle velocity c_σ is defined as,

$$c_\sigma = \frac{d\sigma}{dt} = \frac{1}{h}\left[c_z - c_x\left(\sigma\frac{\partial h}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \left(\sigma\frac{\partial h}{\partial t} + \frac{\partial \eta}{\partial t}\right)\right] \qquad (2)$$

The left hand side of equation (1) describes the free transport of fluid particles and the right hand side of equation (1) is the collision model proposed by Bhatnagar, Gross and Krook (1954), which describes the effect of particle interaction on the distribution function. The moments of the equilibrium distribution function are actually macroscopic water flow quantities we are interested in (Ghidaoui et al. 2001). Since we here extend the original model to the vertical dimension and make use of the σ-transformation, the equilibrium distribution function differs slightly from that in Ghidaoui et al. (2001),

$$q = \frac{1}{2\pi P}\exp\left[-\frac{(c_x-u)^2}{2P}-\frac{(c_z-w)^2}{2P}\right] \tag{3}$$

where $P = p/\rho$ is the pressure, (c_x, c_z) is the particle velocity and (u,w) is the macroscopic water velocity.

With the help of the first-order Chapman-Enskog multi-scale expansions (Vincenti and Kruger 1965), it can be shown that zeroth and first moments of (1) give

$$\frac{\partial h}{\partial t}+\frac{\partial uh}{\partial x}+\frac{\partial \omega h}{\partial \sigma}=0 \tag{4}$$

$$\frac{\partial uh}{\partial t}+\frac{\partial}{\partial x}\left(u^2h+Ph-2\upsilon h\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial \sigma}\left[u\omega h-\left(\sigma\frac{\partial h}{\partial x}+\frac{\partial \eta}{\partial x}\right)P-\frac{\upsilon}{h}\frac{\partial u}{\partial \sigma}\right]=0 \tag{5}$$

where ω is the velocity component in the σ direction. It should be noted that the diffusion terms emerge naturally from the right hand side of equation (1). That is, an algebraic difference between the f and q at the microscopic level becomes a second derivative at the macroscopic level.

3. THE BGK-BASED NUMERICAL MODEL FOR SHALLOW WATER EQUATIONS

The discrete form of equations (4) and (5) in vector forms is

$$\mathbf{W}_{i,k}^{n+1}=\mathbf{W}_{i,k}^{n}-\frac{1}{\Omega_{i,k}}\left(\int_{t_n}^{t_{n+1}}\mathbf{F}_S L_S\,dt\right) \tag{6}$$

where $\Omega_{i,k}$ is the area of the control volume that is centered at point (i,k), L_s is the length of the side s, $\mathbf{W}$ denotes conservative variables and $\mathbf{F}$ represents the flux as a function of t for $t_n \le t \le t_{n+1}$. Since the fluxes are moments of distribution functions at the cell interfaces, an algorithm for the fluxes in (6) is obtained by discretizing the Boltzmann equation and the result is as follows (Liang et al. 2006)

$$\begin{aligned}\int_{t_n}^{t_{n+1}}\mathbf{F}_S L_S\,dt &= \gamma_1(\mathbf{A}_{si}+\mathbf{A}_{so})+\gamma_2\left[\mathbf{n}_s\cdot\nabla(\mathbf{B}_{si}+\mathbf{B}_{so})+\mathbf{t}_s\cdot\nabla(\mathbf{D}_{si}+\mathbf{D}_{so})\right]\\ &+\left(\Delta t-\gamma_1+\gamma_6\frac{\partial}{\partial t}\right)\mathbf{E}_s+\gamma_5\left(\mathbf{n}_s\cdot\nabla\mathbf{G}_s+\mathbf{t}_s\cdot\nabla\mathbf{I}_s\right)+\gamma_3\left(\mathbf{n}_s\cdot\nabla\mathbf{G}_s+\frac{\partial}{\partial t}(\mathbf{E}_{si}+\mathbf{E}_{so})\right)\end{aligned} \tag{7}$$

where $\gamma_0=\exp(\Delta t/\tau)$, $\gamma_1=\tau-\tau\gamma_0$, $\gamma_2=\tau^2+(\tau^2+\tau\Delta t)\gamma_0$, $\gamma_3=\tau^2(1-\gamma_0)$, $\gamma_4=1-\gamma_0$, $\gamma_5=-\tau\Delta t+2\tau^2(1-\gamma_0)-\tau\Delta t\gamma_0$, $\gamma_6=\tau\Delta t-\tau^2(1-\gamma_0)$, and the matrixes **A**, **B**, **D**, **E**, **G**, **t**, and **I** can be found in Liang et al. (2006).

In the algorithm, free surface and vertical velocities are obtained by integration of the divergence-free condition,

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(u\int_0^1 hd\sigma\right)=0 \tag{8}$$

$$\frac{\partial}{\partial x}\left(\int_0^{z_k} udz\right)+w_k-w_0-=0 \tag{9}$$

The Boltzmann-based algorithm involves the following steps:

Step-0: Calculate initial values by the results from the previous time step and Monotone Upstream-centered Schemes for Conservation Laws (MUSCL)-based interpolations;

Step-1: Update x-direction momentum with x-direction flux;

Step-2: Update z-direction momentum by the continuity equation (9);

Step-3: Update surface elevation by equation (8);

Step-4: Update x-direction momentum with σ-direction flux.

4. MODEL VERIFICATIONS AND VALIDATIONS

4.1 Linear waves.

The ability of the BGK model in resolving small-amplitude wave motions is tested in this section. The typical linear wave solutions can be obtained from the first-order asymptotic analysis as follows (Mei, 1989)

$$\omega=\sqrt{ghk}\,,\eta=A\cos(kx-\omega t)\,,u=\frac{A}{h}\sqrt{gh}\cos(kx-\omega t)\,,w=\frac{A}{h}\sqrt{gh}\left[kh\left(1+\frac{z}{h}\right)\right]\sin(kx-\omega t)$$

In the current test, the calm water depth is 1 m, the wave number is 0.01, and the wave amplitude is 0.001 m. The length of the computational domain is five times the wavelength. The flow domain is divided into 3200 sections along the horizontal direction and 10 grid layers along the vertical direction. Initially, water is calm inside the whole domain. The water is set in motion by a sinusoidal forcing at the upstream boundary. At the downstream boundary, periodic conditions are enforced.

It can be seen from Fig. 1 that all modeled quantities such as horizontal velocity, vertical velocity and free surface displacements are well-predicted in terms of both magnitudes and phases. These good agreements fully reflect the ability of the model to resolve the free surface and the horizontal as well as the vertical velocities in the inviscid limit.

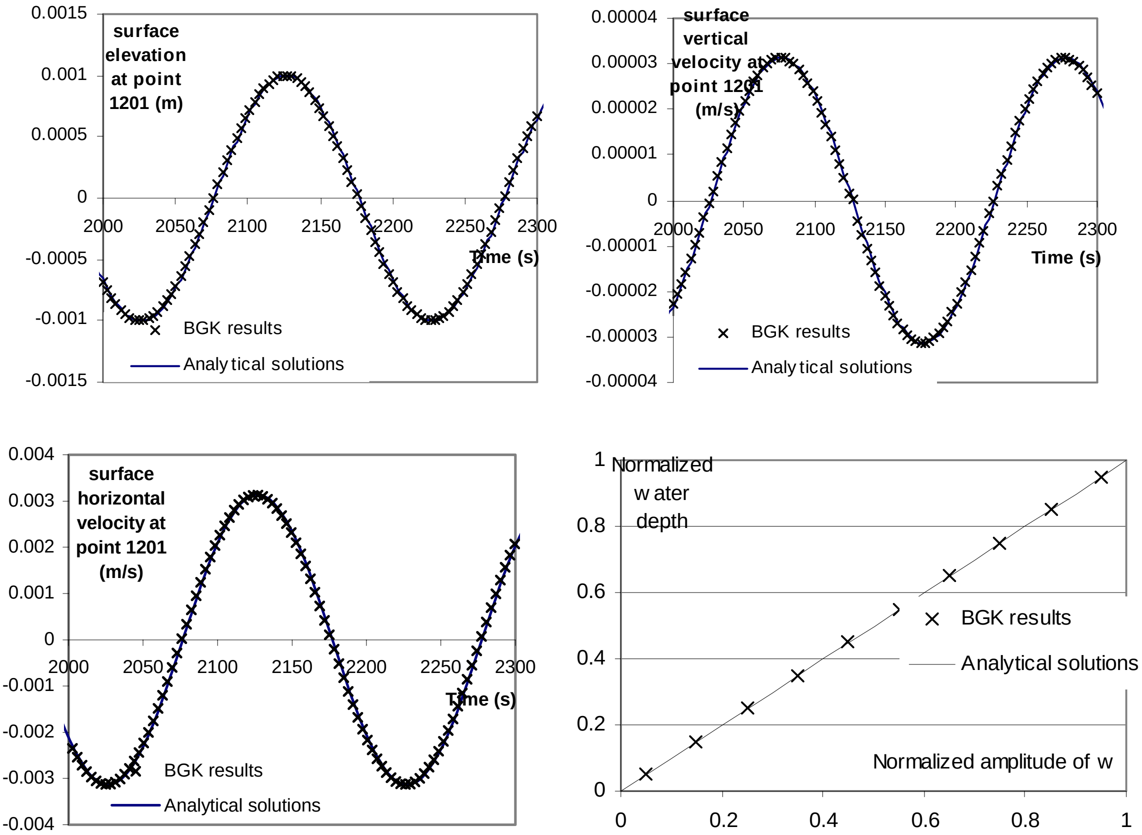

FIGURE. 1 (up left) surface elevation at the water surface at 1200 m from the inflow from 2000 to 2300 s; (up right) surface vertical velocity at the water surface at 1200 m from the inflow from 2000 to 2300 s; (down left) surface horizontal velocity at the water surface at 1200 m from the inflow from 2000 to 2300 s; (down right) Amplitude of vertical velocity.

4.2 Wind-driven currents.

One major attractive features of the proposed model is related to the fact that the Boltzmann-based fluxes comprise both waves and diffusion. The ability of the Boltzmann-based model to resolve diffusive (viscous) phenomena is illustrated in this section using wind-driven flows inside a closed basin. This flow pattern is commonly observed in lakes and reservoirs and greatly influences mixing, transport and waves in the water bodies. The fluid near the water surface is dragged by the wind and produces a windward lowering and a leeward rise of the water level. Subsequently, the pressure gradient due to the tilting of the water surface generates return flows at the bottom of the tank. Thus, the circulation in the vertical plane forms.

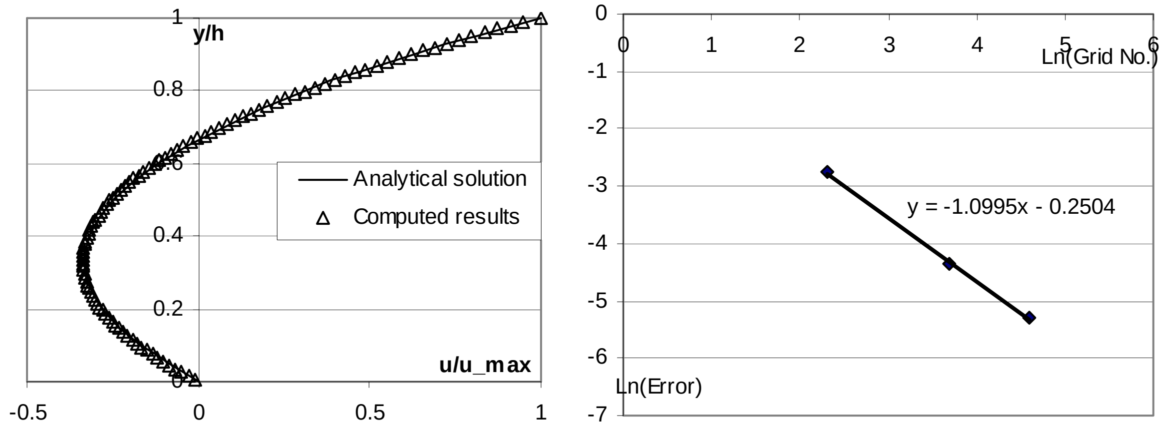

FIGURE. 2. (left) Velocity profile of a wind-driven current with constant eddy viscosity; (right) Convergence study.

For laminar flows, Koutitas and O'Connor (1980) showed that there exists an analytical expression for the velocity,

$$u = \frac{SH\sigma}{4\rho\upsilon_e}(3\sigma - 2)$$

where S is the surface wind stress, H is the total water depth, σ is the nondimensional elevation, ρ is the density of the fluid and υ is the viscosity. Fig. 2(left) shows the computed as well as the analytical velocity distribution. The good agreement illustrates that the Boltzmann-based fluxes can accurately represent the diffusive terms by simply setting the relaxation time equal to the product of viscosity and pressure. Fig. 2(right) presents the rate of convergence of the model and shows that it is of order 1.0 – a result that is consistent with the study by Torrilhon and Xu (2006) for sufficiently fine meshes.

The success of the model in modeling waves and laminar flows shows that the BGK scheme resolves both wave and diffusive phenomena without need for any special treatment such as operator splitting, characteristics decomposition etc. We now illustrate that the scheme is also applicable to turbulent flows by simply setting the relaxation time equal to the product of eddy viscosity obtained from any turbulence model and pressure. This feature is tested by wind-driven turbulent currents. The test rig in this study corresponds to the lab test reported by Tsuruya et al. (1985). The closed basin is 30.48 m, water depth is 0.15 m and wind velocity is 6.9 m/s. Tsanis (1989) showed that the double-logarithmic structures in turbulent wind-driven flows can be well approximated by a parabolic distribution of eddy viscosity as follow,

$$\upsilon_e = \frac{\lambda u_*}{h}(z + z_b)(h - z + z_s)$$

where λ = 0.35, u_* is the surface friction velocity, z_b = $2.2\times10^{-4}h$ and z_s = $0.6\times10^{-4}h$. Computed results are compared with measured ones in Fig. 3(left). The model well captures the surface jet and the return flow at the bottom. The computed mean velocity is in good agreement with the measurements in terms of both shape and magnitude, as shows the ability of the proposed model in modeling turbulent flows.

Another interesting related phenomenon is the seiche due to initiation of wind. The theoretical period of seiche is (Kamphuis 1990)

$$T = \frac{2L}{\sqrt{gh}}$$

where L is the length of the closed basin, h is the water depth. The theoretical period for our case is 50.3 s. A time history of surface displacement at 0.1 L from the lee end is shown in Fig. 3(right). The calculated period is 50.5 s, which is 0.4% different from the analytical period.

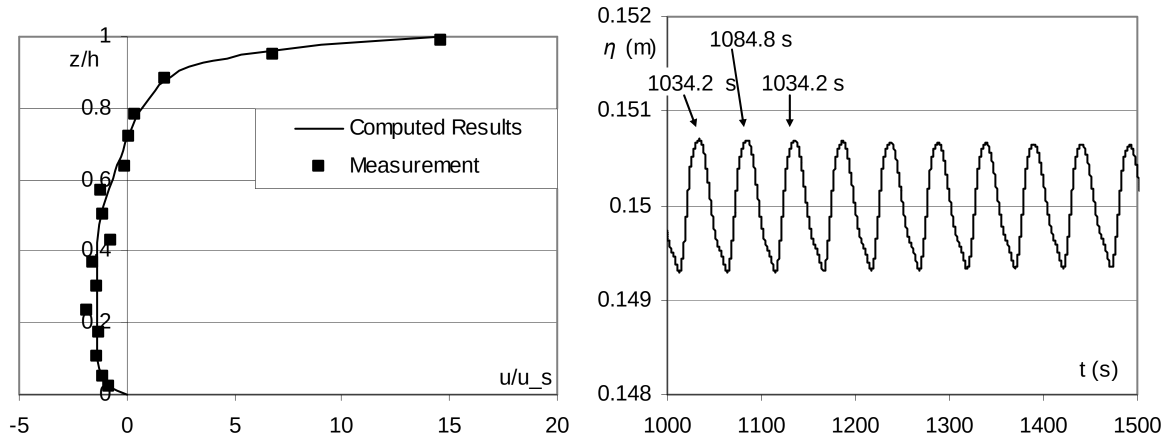

FIGURE. 3 (left) velocity profile of a wind-driven current with parabolic distribution of eddy viscosity; (right) time history of free surface displacement after the initiation of wind

5. CONCLUSION

This paper focuses on illustrating how the Boltzmann theory can be used to formulate numerical algorithms for shallow water equations in the vertical plane. The paper shows that the shallow water equations are obtainable from the moments of the Boltzmann equation and uses this connection to formulate a numerical model for shallow surface water flows in the vertical plane on the basis of the Boltzmann equation. The Boltzmann-based model is capable of modeling the free surface as well as the horizontal and vertical velocities in a variety of surface water problems including linear waves, laminar flows, turbulent flows as well as seiche.

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