INVESTIGATIONS OF WAVE RUNUP USING A LBGK MODELING APPROACH

JANNETTE B. FRANDSEN

Dept. Ocean & resources Engrg., University of Hawaii Manoa,
2540 Dole Street, Holmes Hall 405, Honolulu, HI 96822, USA.
Email: frandsen@hawaii.edu

Abstract

In this paper, the suitability of a LBGK modeling approach is tested to examine the behavior of free surface water waves in shallow water. The free surface model is fully non-linear. Mathematically, this is taking into account through the collision integral. The present LBGK model discretizes the tidal wave equations and approximate the collision between particles using a single time relaxation. It is assumed that the waves do not overturn; limiting the present study to cover small to steep waves. The test case includes a wave run-up study on a sloping beach generated by an initial wave profile which resembles a tsunami. The uniform grid solutions are compared with analytical solutions and other findings reported in the literature. This study is important to a variety of applications, in particular, the coastal engineering community. The present investigations could potentially play a future role in storm surge predictions.

1. INTRODUCTION

The present paper present some preliminary free surface water wave test cases based on single phase Lattice Boltzmann (LB) model approach. The central idea behind proposing the mesoscopic LB formulation is to capture smaller scales naturally, with the long-term goal of predicting breaking wave motion in multiphase flow. The mathematical framework allows for investigations of micro structures in the flow at the mesoscopic level and thus, in theory, offer further insight into underlying mechanism of nonlinear processes at the free surface than the conventional water wave continuum mechanics level models. Herein we are particular interested in assesing/improving the traditional numerical long wave models. These models typically discretize the Non-Linear Shallow Water (NSLW) or the Boussinesq equations. The challenge with long wave models are primarily the handling of shocks at the free surface and the shoreline motion trajectories during wave run-up on sloping beaches. Moreover the numerical long-wave literature often discuss the short-comings of a particular model in relation to the lack of capturing dispersion effects, wave breaking, shoreline motion trajectory and wave transition from intermediate depths to shallow water. These issues have been the motivation of the present paper.

The LB method simulates fluid flow by tracking particle distributions in a Lagrangian manner. The particles are constraint to move on lattices, that is, by default regular lattices. In contrast to continuum based numerical models, where only space and time are discrete, the discrete variable of the LB model are space, time and particle velocity. The Boltzmann equation relates the time evolution and spatial variation of a collection of molecules to a collision operator that describes the interaction of the molecules. Mathematically, the collision integral poses difficulties. This is reflected in the literature through descriptions of models with different levels of accuracy in the approximations of the integral. We consider a model in which the collision assumptions are simplified to a single-time relaxation form. This is the most popular and also considered to be the most simple form of the Lattice Boltzmann equations. It is referred to as the Lattice Bhatnagar-Gross-Krook (LBGK) scheme, after Bhatnagar et al. [1].

Some LBGK investigations showing the solvers ability to predict weak bores and handling of shocks have been carried out [3]. Fairly good agreement were found with Riemann solutions and tank (1×1 m^2) model-scale experiments. We present some preliminary LBGK predictions involving wave run-up on sloping beaches. Compared to the previous investigations, we have introduced a new force term (beach slope) and have a much larger fluid domain (50 Km). We should also note that our initial condition resembles a tsunami. We shall show that good agreement was found at the macroscopic shoreline motion level with NSLW and the Boussinesq equations and the analytical solution of Carrier et al. [2].

2. GOVERNING EQUATIONS

Consider the flow of water with a free surface under gravity in a three-dimensional domain where $x - y$ denotes a horizontal plane whilst z defines the vertical direction. The free surface elevation coincides with the z-axis. Investigation of the behavior of free surface water waves in tanks and open domains with flat beds and uniform sloped beds are undertaken. Assuming viscous flow in shallow water depths and that the vertical component is negligible, the fluid motion is governed by the tidal wave equations,

$$
\begin{gathered}
\frac{\partial h}{\partial t} + \frac{\partial (h\,u_x)}{\partial x} + \frac{\partial (h\,u_y)}{\partial y} = 0\,; \\
\frac{\partial (h\,u_x)}{\partial t} + \frac{\partial (h\,u_x^2)}{\partial x} + \frac{\partial (h\,u_x\,u_y)}{\partial y} = -g\,\frac{\partial}{\partial x}\Big(\frac{h^2}{2}\Big) + h\,\nu\Big(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\Big)\,; \\
\frac{\partial (h\,u_y)}{\partial t} + \frac{\partial (h\,u_y^2)}{\partial y} + \frac{\partial (h\,u_x\,u_y)}{\partial x} = -g\,\frac{\partial}{\partial y}\Big(\frac{h^2}{2}\Big) + h\,\nu\Big(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\Big) - gh\frac{\partial h_b}{\partial y} - \frac{\tau_b}{\rho}\,,
\end{gathered}
\tag{1}
$$

where $h = h_0 + \zeta$, h_0 is the still water depth (or initial water depth) and ζ denotes the free surface elevation measured vertically above still water level, t is time, $g = 9.81$ m/s^2 denotes acceleration due to gravity, $\rho = 1000$ kg/m^3 is the water density, $\nu = 1 \times 10^{-6}$ m^2/s is the kinematic viscosity, and u_x, u_y are the horizontal depth averaged velocity components. The bed slope height is h_b and τ_b denotes the bed shear stress.

3. NUMERICAL MODEL.

A LB single time relaxation model (LBGK) has been extended to simulate free-surface flows in shallow water. The depth averaged tidal wave equations (1) have been discretized. The flow field is represented by particles which follows lattice points which stream and collides when meeting. With reference to the Boltzmann equation of classical kinetic theory, the distribution of fluid molecules is represented by the particle distribution function $f_i(x, y, t)$ where i denotes propagation direction. The function defines the mass density ρ of lattice particles (herein equivalent to a water column h) located at lattice point (x, y) at time t which moves with velocity c_i. At a lattice point the particles can have different velocities following a Maxwell distribution of equilibrium. The single phase LBGK formulation for shallow liquid flows on sloped beds can be written as

$$\frac{\partial f_i}{\partial t} + c_i \nabla f_i = -\frac{(f_i - f_i^{eq})}{\tau} - \frac{c_i \Delta_t}{6 c^2} g h \frac{\partial h_b}{\partial x}. \tag{2}$$

The first term represents the effect of the propagating fluid motion and the second term describes the convection. The first term on the right-hand-side (RHS) is the non-equilibrium distribution function which describes the effect of collisions (Succi [7]). The second term on the RHS represent force terms. Herein we have one force term which represent the bed slope. The time-scale parameter τ describe the collisional relaxation to the local equilibrium. In present formulation it is limited to a single value and is defined as $\tau = 0.5 + 3(\nu \Delta t)/\Delta x^2$ where ν is the kinematic viscosity of water. To ensure numerical stability $\tau > 0.5$. The equilibrium distribution functions f_i^{eq} represents the invariant function under collision (no gradients are involved) and is dependent of the microscopic velocity vector c_i. It is essential that the equilibrium distribution functions satisfy the Navier Stokes equation. These functions can be derived from the Boltzmann equation using a Chapman-Enskog expansion. Details of the Chapman-Enskog method for the classical Boltzmann equation can be found in Gombosi [4]. The LBGK based f_i^{eq} in the (x-y)-plane, in shallow water [6], expressed as follows,

$$\begin{aligned}
f_{i=0}^{eq} &= h_0 + \zeta - \frac{5g(h_0+\zeta)^2 \Delta t^2}{6\Delta x^2} - \frac{2(h_0+\zeta)\Delta t^2}{3\Delta x^2}(u_i\, u_i) \;, \\
f_{i=1-4}^{eq} &= \frac{g(h_0+\zeta)^2 \Delta t^2}{6\Delta x^2} + \frac{(h_0+\zeta)\Delta t^2}{3\Delta x^2}(c_i\, u_i) \\
&+ \frac{(h_0+\zeta)\Delta t^4}{2\Delta x^4}(c_i\, c_j\, u_i\, u_j) - \frac{(h_0+\zeta)\Delta t^2}{6\Delta x^2}(u_i\, u_i) \;, \\
f_{i=5-8}^{eq} &= \frac{g(h_0+\zeta)^2 \Delta t^2}{24\Delta x^2} + \frac{(h_0+\zeta)\Delta t^2}{12\Delta x^2}(c_i\, u_i) \\
&+ \frac{(h_0+\zeta)\Delta t^4}{8\Delta x^4}(c_i\, c_j\, u_i\, u_j) - \frac{(h_0+\zeta)\Delta t^2}{24\Delta x^2}(u_i\, u_i) \;,
\end{aligned} \tag{3}$$

where the subscript i denotes the propagating direction of the discrete velocities vectors (here D_2Q_9) given by

$$c_{i=1-4} = (cos[(i-1)\pi/2], sin[(i-1)\pi/2])c \quad \text{and}$$
$$c_{i=5-8} = (cos[(2i-9)\pi/4], sin[(2i-9)\pi/4])\sqrt{2}c\,, \tag{4}$$

where the lattice velocity $c = \Delta x/\Delta t_x = \Delta y/\Delta t_y$ where Δx is the lattice size (uniform) and Δt is the duration of the time step. The rest particle correspond to i=0 ($c_{i=0} = 0$) and others represent lattice vectors in the direction of the nearest neighbors.

The present 1-D LBGK model does not include the conventional dynamic and kinematic boundary conditions at the free surface. Instead, the free-surface dynamics are accounted for through the non-equilibrium particle distribution function ($f^{neq} = f_i - f_i^{eq}$). No additional algorithm or surface boundary conditions are necessary to be prescribed. It is essential to satisfy the constraints of the hydrodynamics moments at all times; the LB solvers equivalent means of conserving mass and momentum. We should stress that the present model can only expect to work well for a continuous surface in shallow water. Discontinuous surfaces and/or other water depths would involve entirely new LB formulations. In particular, new sets of equilibrium distribution functions would be needed. A numerical Boltzmann formulation in arbitrary water depth could include a free surface treatment as known from traditional solvers, e.g., a volume of fluid algorithm, a level set, etc. to account for breaking. Therefore a host of models could potentially be developed to bridge the gap between micro and macroscopic models and thus potentially advance the way we currently model free surface flows.

The LB equation (2) is solved by first calculating f_i^{eq}. Then, the discretized LB equation is separated in into advection and diffusive parts and f_i computed. Herein we have used a first order time integration scheme. Finally, the macroscopic variables of the free surface (ζ) and the depth averaged velocities (u_x, u_y) are calculated as the first and second moments of the distribution function,

$$\zeta = \sum_i f_i - h_0 \quad \text{and} \quad u_x = \frac{1}{(h_0+\zeta)}\sum_i c_{ix}\, f_i \quad \text{and} \quad u_y = \frac{1}{(h_0+\zeta)}\sum_i c_{iy}\, f_i\,. \tag{5}$$

The procedure is repeated at each time step Δt. The hydrodynamic moments of the equilibrium distribution functions are satisfied at every time step. Further, it should be noted that it is assumed that the Mach no. Ma=u/c_s <<1 where c_s is the speed of sound. The Peclet no. Pe=$u\,\Delta x/\nu < 2$ and the Courant no. Cr=$u\,\Delta t/\Delta x < 1$ are also obeyed ($u = u_x$ or $u = u_y$).

4. CASE STUDY

In the following test case we are concerned with testing the solvers ability to handle wave run-up on beaches. The predictions of the shoreline trajectory represent a classical bench mark test of numerical models, especially because of the challenge of accurately predicting the wave motion when the depths are vanishing into dry-states. The present

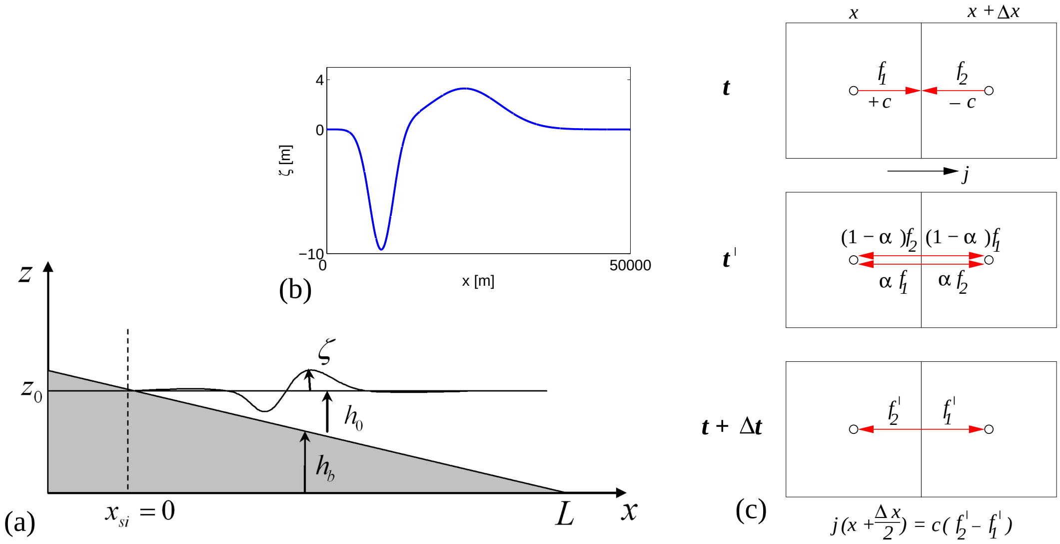

FIGURE 1. (a) Definition sketch of wave run-up study. (b) Initial wave profile. (c) D_1Q_3 model.

test case represents a tsunami wave generated run-up/run-down study on a beach. The initial shoreline location is located 50,000 m from starting point of the slope and $z_0 =$ 5000 m. The slope of the beach is constant $\frac{\partial h_b}{\partial x} = 1/10$ and uniform and span the whole computational domain, as shown in Fig. 1(a).

The present numerical model have simplified the $(x - y)$ plane of discrete microscopic velocity model D_2Q_9 further and assumed that at any time, the LB fluid is characterized by the populations of the three discrete microscopic velocity model D_1Q_3 (Fig. 1c); representing a 1-D model. Initially the velocity in the flow domain is zero and the free surface described by the form of a leading depression N-wave shape, typically caused by an offshore submarine landslide,

$$\zeta = a_1\, e^{\left(-k_1(x-x_1)\right)} - a_2\, e^{\left(-k_2(x-x_2)\right)} \quad \text{and} \quad u = 0 \quad \text{at t=0}\,, \tag{6}$$

where $a_1 = \frac{1}{3}a_2$=0.006, a_2=0.018, $k_1 = \frac{1}{9}k_2$=0.4444, x_1=4.1209 and x_2=1.6384, after Carrier et al. [2]. The initial wave profile is also shown in Fig. 1(b). We have applied the initial wave profile (6) at a distance of 50,000 m offshore measured from the estimated initial shoreline position x_{si} to the maximum water surface depression of approximate 9 m. At x=L, the distribution functions is $f_1 = f_2$ following the D_1Q_3 notation (Fig. 1c). In addition, a no flux condition on macroscopic variables h and u have been prescribed ($h(L) = h_0(L) =$ 5000 m and $u(L)$=0), to suppress reflections. In dry areas ($h = 0$), a thin layer of fluid ($h = 10^{-5}$ m) is prescribed. The shoreline is defined as the first point where $h > 10^{-5}$ m. Other than this, no special treatment of the shoreline motion has been prescribed herein, as the LB solver then handles the moving interface between water

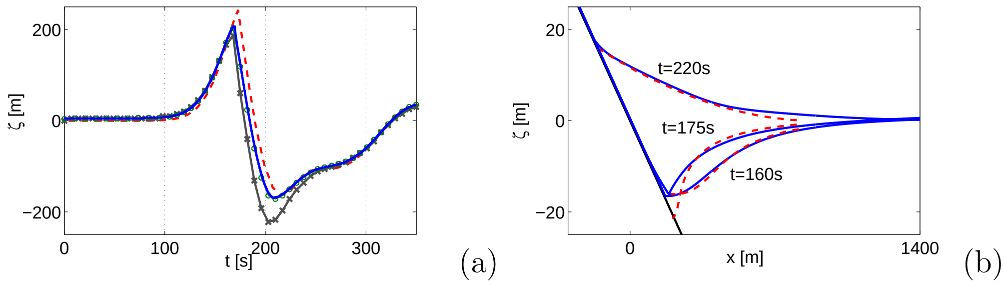

FIGURE 2. Shoreline location movement. − −, semi-analytical solution [2]; LB solutions: −x−, 10,000 nodes; —, 30,000 nodes; − ∘ −, 40,000 nodes.

TABLE 1. LB results versus case "d" of Carrier et al. [2].

Result	Lattice Boltzmann (30,000 nodes)	Semi-Analytical (Carrier et al., 2003)
Maximum run-up	-170 m	-164 m
Maximum draw-down	+207 m	+242 m

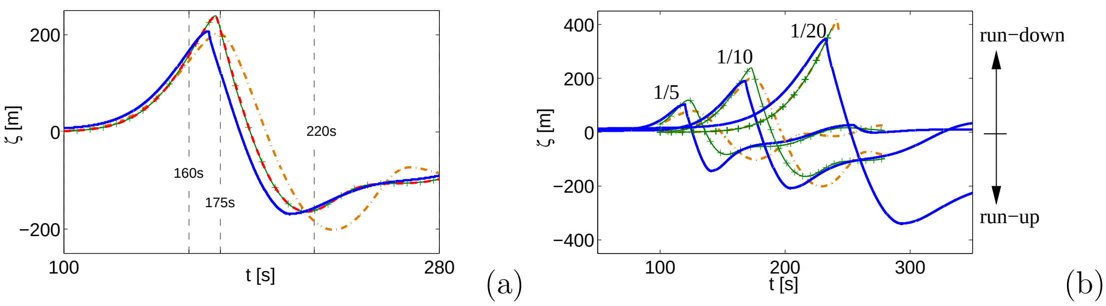

FIGURE 3. Comparison of shoreline location movement. —, LB; − −, semi-analytical [2]; −+−, NLSW [5]; − · −, Boussinesq [5]. (a) slope: 1/10. (b) slope variation.

and land automatically. Some improvement on this issue should, however, be considered in future tests, as discussed later. At the outlet ($x = -500$ m), no boundary conditions are necessary to be introduced as the present wave run-up test case does not exceed the length of the slope. We have further assumed that the bed shear stress is negligible.

First, simulations were carried out for three grid resolutions (uniformly distributed cells), 10,000, to 60,000 nodes which include $\Delta x = 5.05$ to 0.84 m, respectively. The corresponding time step is $\Delta t = 0.03$ s. Fig. 2(a) displays the shoreline motion for the different LB grid solutions relatively to the analytical reference solution of [2]. The 30,000 and 40,000 nodes resolutions yielded similar results. Therefore, the following results will be based on the 30,000 1-D model resolution. Fig. 2(b) shows snapshots of the free surface profiles at three selected time instances at which the LB 30,000 solution is compared with the analytical solution. The results of Carrier et al. [2] and the LBGK solutions are also outlined in Table 1. The LB solution compares fairly well except at $t = 175$ s, the maximum run-down instance of the analytical solution. The LB solution underestimate the run-down and the run-up is overestimated. The maximum run-down and run-up of the LB solution also occur slightly earlier than the analytical predicted ones. We would expect the inclusion of the viscosity in the LB solution would results in some reduction of the draw-down and an increase in the run-up. However, it was observed that the LB velocity is in general underestimated and thus this could contribute further to the discrepancy. We are currently addressing the issue. Furthermore, we have also compared the LB solutions with the NLSW and Boussinesq solutions produced by Pedersen [5], as shown in Fig. 3(a). These models are based on a time step of 0.09 s and about 3171 nodes uniformly distributed. The NLSW solution [5] compares well with the analytical solution whereas the Boussinesq model predict run-down similar to the LB model. The Boussinesq model run-up is, however, over-predicted and the occurrence of maximum run-down and run-up is delayed compared to the other models.

We also tried to study the effect on run-up/run-down when varying the slope (Fig. 3b). The steepest slope case (1/5) shows smallest run-up/run-down, as expected. The grid convergence study of Pedersen [5] showed no run-up when $\frac{\partial h_b}{\partial x} = 1/20$ whereas the LB model did predict shoreline motion profile similar to the 1/10 case (just enlarged). The other beach slope cases showed a similar effect on the shoreline motion when comparing the LB with the NLSW and Boussinesq solutions [5]. Improvement in terms of higher time integration scheme of the LB solutions are expected to decrease the discrepancy observed.

5. FINAL REMARKS

The present Boltzmann modeling approach may be a new potentially competitively candidate for wave run-up an inundation modeling. Obviously unstructured lattices and a higher order time integration scheme are highly desirable for the run-up/run-down problem. These issues and the lack of resolution in the near-shore region could be a simple explanations for not achieving better agreement. There are however several other fundamental issues which need further investigations. First the numerical operator splitting scheme should be questioned (avoided), and second, an improvement of the accuracy of the approximation of the collision integral should be tested through a multi-relaxation-time scheme.

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