A FAMILY OF SOLUTIONS FOR THE ONE-DIMENSIONAL TRANSPORT EQUATION

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Abstract

We give herein a family of functions that satisfies the one-dimensional convection-diffusion (transport) equation with constant coefficients. A preliminary analysis and numerical examples are provided.

1. INTRODUCTION AND BACKGROUND

Convection-diffusion differential equations are used extensively to study physical processes in the sciences and engineering, including the modeling of subsurface contaminant transport. The numerical solution of such equations can, however, be plagued by spurious (and physically unmeaningful) oscillations, particularly when convection is the dominant process. To ameliorate the effect of these oscillations, the technique of "upstream weighting" is often applied to the convective term. This introduces artificial dispersion, often at the expense of "smearing" the sharp solution profile that characterizes the convection-dominated problem. The goal, therefore, is to obtain highly accurate solutions that suffer from neither oscillations nor "smearing."

In this paper, we provide a formula for a family of functions that satisfies the convection-diffusion equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - v\frac{\partial c}{\partial x}. \tag{1}$$

Here the diffusion coefficient D and velocity coefficient v are both positive constants.

Since every member of the family of functions is a solution of (1), so is every piecewise and/or linear combination of them. Thus the challenge lies in selecting the values of certain free parameters in such a way as to satisfy initial and boundary conditions. The primary advantage of this approach is that the issues of spurious oscillations and "smearing" do not arise.

Since we have just begun work on this problem, this paper will be brief and should be considered in the framework of this context. We will give below the family of solution functions, a preliminary analysis of some of the parameters that arise, and a computational example that reveals the potential promise of this approach.

2. THE FAMILY OF SOLUTION FUNCTIONS

It is completely straightforward to prove the following:

Theorem. Any function of the following form is a solution of (1):

$$c(x,t) = p + q\exp(k(x + \alpha vt)), \tag{2}$$

where the constants p, q, and α are arbitrary and where the constant k is defined

$$k = \frac{v}{D}(\alpha + 1).$$

3. PRELIMINARY ANALYSIS

Consider the differential equation (1) with the initial condition (for $x > 0$)

$$c(x, 0) = 0 \tag{3}$$

and the boundary conditions (for $t > 0$)

$$\begin{aligned} c(0, t) &= 1 \\ c(\infty, t) &= 0. \end{aligned}$$

This problem has the "analytical" solution [Bruggeman, 1999]

$$c(x, t) = \frac{1}{2}\left\{\operatorname{erfc}\left(\frac{x - vt}{2\sqrt{Dt}}\right) + \exp\left(\frac{xv}{D}\right)\operatorname{erfc}\left(\frac{x + vt}{2\sqrt{Dt}}\right)\right\}. \tag{4}$$

For the values $v = 10$, $d = 2$, and $t = 1$, the solution (4) is depicted in Figure 1.

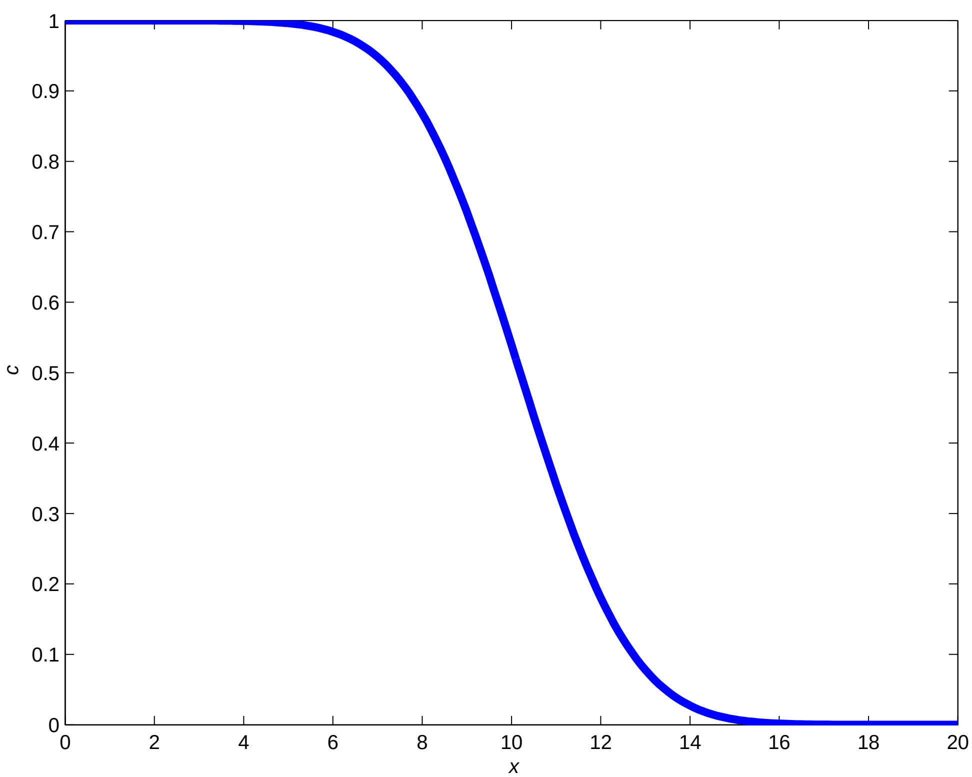

FIGURE 1. Analytical solution (4) for $v = 10$, $d = 2$, and $t = 1$

With reference to our solution (2), it is clear that a single function of this form cannot hope to capture a solution that resembles the curve in Figure 1. However, a piecewise function of the form

$$c(x,t) = \begin{cases} c_1(x,t) = p_1 + q_1 \exp(k_1(x + \alpha_1 vt)) & \text{if } 0 \le x \le X \\ c_2(x,t) = p_2 + q_2 \exp(k_2(x + \alpha_2 vt)) & \text{if } X \le x \end{cases} \tag{5}$$

might do the job quite admirably.

We now examine what conditions on the parameters p_1, p_2, q_1, q_2, α_1, α_2 and X are necessary to ensure that (5) can capture the solution profile in Figure 1.

First, consider the function $c_2(x,t)$ in (5). It is clear, as x becomes large, that we wish $c_2(x,t)$ to decay to 0. Thus we require that $p_2 = 0$ and that $k_2 < 0$, which in turn requires that $\alpha_2 < -1$. Similarly, $c_1(x,t)$ requires a growing exponential as x increases, which stipulates that $k_1 > 0$ and thus that $\alpha_1 > -1$.

Other interesting questions arise with respect to the transition point X. Because the solution (4) is smooth, it is reasonable to enforce that both $c(x,t)$ in (5) and its derivative with respect to x are continuous at $x = X$. It may be advantageous to also enforce continuity of $\frac{\partial^2 c}{\partial x^2}$ at $x = X$. However, the question of a good way to choose X is not yet considered.

4. COMPUTATIONAL EXAMPLE

In this section, we provide an example showing that the piecewise function defined in (5) can, with judicious choices of the parameters, produce a solution curve that does a reasonable job of capturing the solution profile given in (4) and depicted in Figure 1. We do confess that the curve we produce is obtained by interpolating (4) at various locations.

First, we choose the transition point $X = 10.18728408$ so as to coincide with the location of the inflection point in the curve in Figure 1. Given this value of X, we then enforce that $c_1(x,1)$ in (5) agrees with $c(x,1)$ in (4) at $x = 0, 7$, and X. These conditions provide the values $p_1 = 1.000365908$, $q_1 = -0.1600446208$, and $\alpha_1 = -0.8583063929$.

Next, we enforce that $c_2(x,1)$ in (5) agrees with $c(x,1)$ in (4) at $x = X$ and 13. This provides two distinct solutions for the pair q_2 and α_2. We choose the the solution pair $q_2 = 0.2365050590$ and $\alpha_2 = -1.132357088$ (along with $p_2 = 0$, as discussed above).

Using these parameter values, we compare the graphs of the solutions computed from (4) and (5) in Figure 2. It is evident that the approximate solution (5) does quite a reasonable job of replicating the exact solution (4).

5. SUMMARY AND FUTURE DIRECTIONS

In this article we have introduced a family of functions that satisfies the differential equation (1), provided a brief analysis of some of the pertinent parameters, and gave an example that reveals the potential of using such functions to solve (1). However, much work remains before this approach is viable as a technique for solving (1). The most difficult challenge appears to be in incorporating the initial conditions (3).

Another potential approach uses the observation that functions of the form

$$c(x,t) = \beta(x - vt) \tag{6}$$

(where β is a constant) also satisfy (1). This observation may prove useful as solution curves, such as those that appear in Figure 1, appear to have parts that are almost linear in shape. This suggests that defining $c(x,t)$ to have both linear and exponential pieces

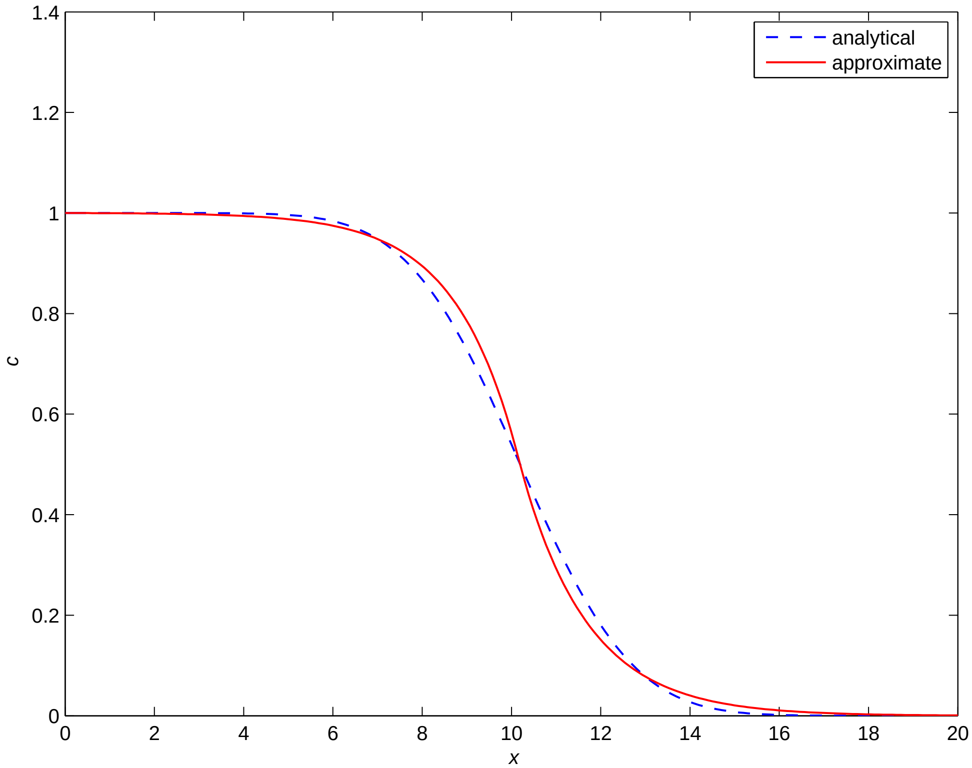

FIGURE 2. Analytical (4) and approximate (5) solutions for $v = 10$, $d = 2$, $t = 1$, $p_1 = 1.000365908$, $q_1 = -0.1600446208$, $\alpha_1 = -0.8583063929$, $p_2 = 0$, $q_2 = 0.2365050590$ and $\alpha_2 = -1.132357088$

may be a fruitful strategy. Another possibility is to have pieces of $c(x, t)$ be composed of linear combinations of functions of the form (2) and (6).

We hope that we may discover techniques that allow functions of the form (2) to be used to provide highly accurate approximations to solutions of (1). After achieving this goal, we will study more general problems in which the coefficients v and D are not constant, as well as extending such results to the multidimensional versions of (1).

REFERENCES

Bruggeman, 1999. Bruggeman, G.A. (1999), Analytical Solutions of Geohydrological Problems, Elsevier, Amsterdam.