A DISCONTINUOUS GALERKIN METHOD FOR THE ONE DIMENSIONAL SHALLOW WATER EQUATIONS INVOLVING BATHYMETRIC TERMS AND DRY AREAS

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Abstract

We develop a Runge-Kutta Discontinuous Galerkin Method to approximate the one dimensional Shallow Water Equations. We introduce a *Flux Modification* technique to preserve steady states at rest with variable bathymetry and a *Slope Modification* technique to deal with dry areas. Numerical results are presented to illustrate the performance of the proposed methods in various test cases.

1. GOVERNING EQUATIONS

In many hydrodynamic applications, the vertical dimension of the problem is negligible compared to its length scale. In this case, one can use the *hydrostatic assumption* to study the flow. Then, if the fluid is supposed to be inviscid, the flow is governed by the so-called Shallow Water Equations. This set of nonlinear equations is of hyperbolic type and can be written in one space dimension as follows :

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial t}\begin{pmatrix} \zeta \\ q \end{pmatrix} + \dfrac{\partial}{\partial x}\begin{pmatrix} q \\ \dfrac{q^2}{\zeta} + \dfrac{g}{2}\,\zeta^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -g\,\zeta\,S_0 \end{pmatrix} \quad \text{in }]0,\ell[\,\times\,]0,T[, \\ \\ \text{Initial and Boundary Conditions,} \end{array} \right. \tag{1}$$

where $\ell > 0$ is the domain length, $T > 0$ the simulation time, g the gravity acceleration, ζ the water depth, q the discharge, $S_0 := \dfrac{\partial b}{\partial x}$ the bed slope and $b\;:\,]0,\ell[\longrightarrow I\!R$ the so-called bathymetry.

2. RUNGE-KUTTA DISCONTINUOUS GALERKIN METHOD

2.1. **Space discretization.** The starting point of our method is the discretization of (1) derived in [Eskilsson, 2004].

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We consider a mesh of the flow domain $]0,\ell[$ which consists of N open intervals $\{I_n\}_{n\in\{1,\dots,N\}} := \{]x_{n-\frac{1}{2}}\ ,\ x_{n+\frac{1}{2}}[\}_{n\in\{1,\dots,N\}}$ where $\{\ x_{n+\frac{1}{2}}\ \}_{n=0}^{N}$ denote ordered points of $]0,\ell[$. Moreover, we introduce the mesh size $h := \max_{n\in\{1,\dots,N\}} \mid I_n \mid$. For a non-negative integer p, the space $\mathbb{P}^p(I_n), n \in \{1,\dots,N\}$, consists of polynomial functions on I_n of degree p at most and we define

$$\mathbb{P}_h^p \ := \ \{\ v\ :\ ]0,\ell[\longrightarrow I\!R \ \text{ such that } \ v|_{I_n} \in \ \mathbb{P}^p(I_n),\ \forall\ n \in \{1,\dots,N\}\}\ .$$

Multiplying the first equation of (1) by $v_h \in [\mathbb{P}^p(I_n)]^2$, integrating over I_n, $n \in \{1,\dots,N\}$, and applying Green's formula, we obtain the following approximation of (1) : Find $\begin{pmatrix} \zeta_h \\ q_h \end{pmatrix} \in C^1([0,T],[\mathbb{P}_h^p]^2)$ such that, $\forall v_h \in [\mathbb{P}^p(I_n)]^2, \forall n \in \{1,\dots,N\}, \forall t \in]0,T[$,

$$\begin{cases} \displaystyle\int_{I_n} v_h\, \frac{\partial}{\partial t}\begin{pmatrix} \zeta_h \\ q_h \end{pmatrix} + \int_{I_n} \frac{\partial v_h}{\partial x} \begin{pmatrix} q_h \\ \dfrac{q_h^2}{\zeta_h} + \dfrac{g}{2}\,\zeta_h^2 \end{pmatrix} \\ \qquad + v_h(x_{n+\frac{1}{2}})\ \mathbb{F}_{n,n+\frac{1}{2}}(\zeta_h, q_h) \ - \ v_h(x_{n-\frac{1}{2}})\ \mathbb{F}_{n,n-\frac{1}{2}}(\zeta_h, q_h) \\ \qquad = \displaystyle\int_{I_n} v_h \begin{pmatrix} 0 \\ -g\ \zeta_h\ S_{0h} \end{pmatrix}, \\ \text{Initial and Boundary Conditions,} \end{cases} \tag{2}$$

where $S_{0h} := \dfrac{\partial b_h}{\partial x}$ is the approximate bed slope defined locally on each mesh cell, b_h is the L^2-projection of b into $\mathbb{P}_h^p$ and $\{\ \mathbb{F}_{n,n\pm\frac{1}{2}}(\cdot,\cdot)\ \}_{n\in\{1,\dots,N\}}$ refer to the so-called *Numerical Flux* functions. We will use the *Harten/Lax/Van-Leer/Einfeldt* flux (HLLE) which has been initially developed for the Euler Equations [Einfeldt, 1988].

2.2. **Time discretization.** The discretization of the time derivative in (2) is performed in an explicit way. In order to ensure an equal accuracy in space and time, we use a Runge-Kutta explicit scheme of order $(p+1)$. If we define the k-th iterate by $\overrightarrow{W}^k := \begin{pmatrix} \zeta^k \\ q^k \end{pmatrix}$, $k \in \mathbb{N}$, the Runge-Kutta scheme of order $q \in \mathbb{N}$ can be written as follows

$$\overrightarrow{W}^{k+1} \ = \ \overrightarrow{W}^k \ + \ \sum_{i=1}^{q} c_i^q\ \mathcal{H}(\overrightarrow{W}_i^k)\ , \tag{3}$$

where $\{c_i^q\}_{i\in\{1,\dots,q\}} \in I\!R^q$, $\{\ \overrightarrow{W}_i^k\ \}_{i\in\{1,\dots,q\}}$ are sub-iterates and $\mathcal{H}$ is an operator defined by (2).

2.3. **Slope limiting.** To avoid spurious oscillations near shocks, slope limiting is necessary. Slope limiting consists in replacing the second term of the right hand side of (3) by

$$\Lambda\Pi(\mathcal{H}(\overrightarrow{W}_i^k)),\ \forall\ i \in \{1,\dots q\},\ \forall\ k \in \mathbb{N}\ , \tag{4}$$

where $\Lambda\Pi(\cdot)$ is an operator which firstly *detects* shocks and then reconstructs the slope of the approximation if necessary. The slope limiting process we use is that detailed in [Cockburn, 2001] and, following the ideas of [Schwanenberg, 2000], we limit the slope of the water height (i.e. $\zeta_h + b_h$). To detect shocks, we use the criterion given in [Krivodonova, 2004]. To do so, we define the quantity

$$\mathcal{I}_n^{\pm} := \frac{(\zeta_h|_{I_n} - \zeta_h|_{I_{n\pm 1}})(x_{n\pm\frac{1}{2}})\ \Theta(\pm q_h|_{I_n})}{|I_n|^{(p+1)/2}\ \|\zeta_h\|_{I_n}} \ , \ \forall\ n \in \{1,\ldots,N\} \ , \tag{5}$$

where $\Theta(s) := 1$ if $s \geq 0$ and 0 otherwise. Then, if $|I_n^+| + |I_n^-| \geq 1$, we apply slope limiting on I_n.

3. TREATMENT OF THE BATHYMETRY : FLUX MODIFICATION

In this section, we will reconsider the scheme (2). The goal (which is identical to that encountered in the Finite Volume Method) is to preserve the *flow at rest* (i.e. $\zeta + b = C$ and $q = 0$). The scheme (2) cannot preserve the flow at rest because such a property requires a compatibility between the numerical flux and the discretization of the bathymetric term. Consequently, we propose to adapt tools from the Finite Volume Methods namely the idea of *well-balanced schemes* [Audusse, 2004]. More precisely, we modify the numerical flux function of (2) by adding a quantity which depends on the bathymetry.

We define the new numerical flux function as follows :

$$\mathbb{F}^*_{n,n\pm\frac{1}{2}}(\zeta_h, q_h, b_h) = \mathbb{F}_{n,n\pm\frac{1}{2}}(\zeta_h^{*,n}, q_h^{*,n}) \pm \delta_n^{\pm}(\zeta_h, q_h, b_h) \ , \ \forall\ n \in \{1,\ldots,N\} \ , \tag{6}$$

where

- $q_h^{*,n} := \zeta_h^{*,n}\ q_h \ /\ \zeta_h$,
- $\zeta_h^{*,n}|_{I_n}(x_{n\pm\frac{1}{2}}) := \zeta_h|_{I_n}(x_{n\pm\frac{1}{2}}) - \max(b_h|_{I_{n\pm 1}}(x_{n\pm\frac{1}{2}}) - b_h|_{I_n}(x_{n\pm\frac{1}{2}}) \ , \ 0)$,
- $\zeta_h^{*,n}|_{I_{n\pm 1}}(x_{n\pm\frac{1}{2}}) := \zeta_h|_{I_{n\pm 1}}(x_{n\pm\frac{1}{2}}) - \max(-b_h|_{I_{n\pm 1}}(x_{n\pm\frac{1}{2}}) + b_h|_{I_n}(x_{n\pm\frac{1}{2}}) \ , \ 0)$,
- $\delta_n^{\pm}(\zeta_h, q_h, b_h) := \begin{pmatrix} 0 \\ \frac{g}{2}\ (\zeta_h|_{I_n}(x_{n\pm\frac{1}{2}})^2 - \zeta_h^*|_{I_n}(x_{n\pm\frac{1}{2}})^2) \end{pmatrix}$.

The following proposition holds

Proposition 1. *The scheme (2) associated with the flux (6) preserves flows at rest, i.e.,* $\forall\ k \in \mathbb{N}$,

$$\left(\ \zeta_h^{(k)} + b_h \equiv C \ \textit{and}\ q_h^{(k)} \equiv 0\ \right) \ \Rightarrow\ \left(\ \zeta_h^{(k+1)} + b_h \equiv C \ \textit{and}\ q_h^{(k+1)} \equiv 0\ \right) ,$$

where C is a fixed positive constant and where exponents refer to the time discretization.

4. TREATMENT OF DRY AREAS : SLOPE MODIFICATION

One major problem to treat dry areas is the following : let (ζ_h, q_h) be the approximate solution (we omit time dependence for shortness). If ζ_h has "small" values, one can obtain a negative water depth during the computation. Besides the non-physical meaning of such values, this poses difficulties in the flux computation since the quantity $\sqrt{g\ \zeta_h}$ appears in the wave speed. To solve this problem, we propose to modify the slope of ζ_h on the elements of the mesh where there are negative values. For the degree $p = 1$, the process splits up as follows :

- If the average of ζ_h is negative on the element, we set $\zeta_h = q_h := 0$ on the element.
- If the average of ζ_h is positive, this implies that ζ_h is negative at only one of the extremities of the element which is denoted by s^-. We keep the average of ζ_h but modify its slope in such a way that ζ_h vanishes at s^-. Then, the discharge q_h is modified by only setting its value at s^- to zero. This process preserves the mass on the element but not the average of the discharge.

The numerical tests presented below indicate that this slope modification procedure is stable and accurate.

Remark 1. *Slope limiting is not applied in the vicinity of an element where there are negative values of the water depth because the slope modification procedure can activate artificially the shock detector.*

5. NUMERICAL TESTS

5.1. **Flow at rest.** The purpose of this test is to verify that a flow at rest is preserved by the flux modification technique. We set $g = 1$, $b(x) = \frac{1}{2}\exp(-100\,(x - 0.5)^2)$, $\ell = 1$, $T = 2$, $N = 16$ and $(\zeta_0\ ,\ q_0) = (2 - b\ ,\ 0)$. Figure 1 shows that the flow at rest is well preserved for various polynomial degrees. In this figure, dotted lines represent the mesh, bold solid lines the L^2-projection b_h of b, and thin solid lines the approximate water height.

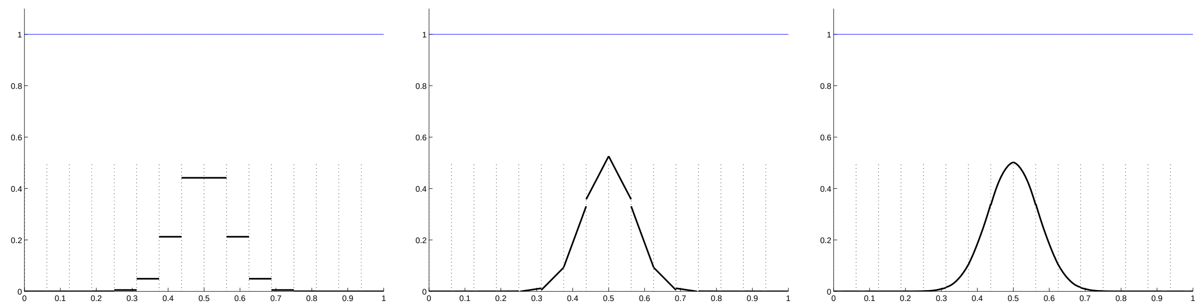

Figure 1: Approximation of the water height at final time for $p = 0$ (left), $p = 1$ (middle), $p = 2$ (right) and $N = 16$

5.2. **Subcritical flow** [Goutal, 1997]**.** The goal of this test is to assess the accuracy of the method. The parameters are $g = 9.81$, $b(x) := \max\{0.2 - 0.05\,(x - 10)^2\ ,\ 0\}$, $\ell = 25$, $T = 100$ and $(\zeta_0\ ,\ q_0) = (2 - b\ ,\ 0)$. A discharge $q_{in} = 4.42$ at the left boundary and a water height $h_{out} = 2$ at the right boundary are imposed. Figure 2 shows the evolution of the approximate solution for $N = 50$ and $p = 1$. Figure 3 presents the error measured in the L^2-norm for various polynomial degrees. Optimal orders of convergence are observed.

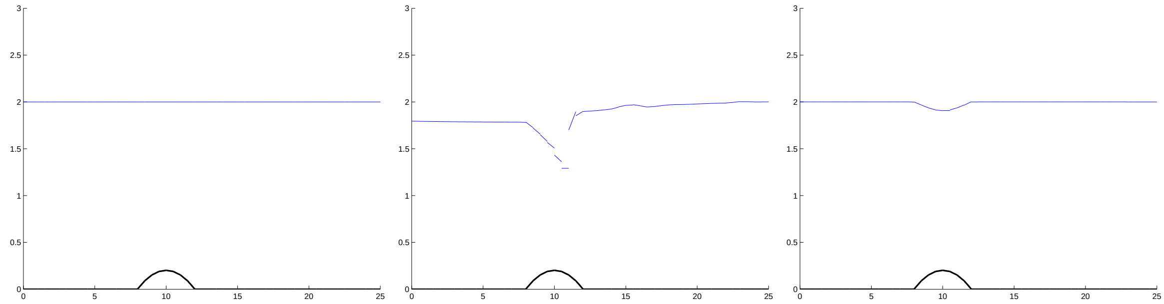

Figure 2: Approximate water height at times $t = 0,\ 25$ and 200 $(N = 50,\ p = 1)$

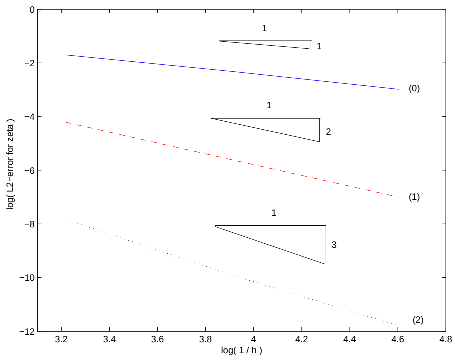

Figure 3: $\log(\|\zeta - \zeta_h\|_{0,]0,\ell[,t=T})$ versus $\log(N)$ for $p = 0,\ 1$ and 2

5.3. **Transcritical flow with shock** [Goutal, 1997]**.** The goal of this test is to evaluate the behavior of the method in the case where the stationary exact solution presents a shock. We set $g = 9.81$, $b(x) := \max\{0.2 - 0.05\,(x - 10)^2\ ,\ 0\}$, $\ell = 25$, $T = 200$, $(\zeta_0\ ,\ q_0) = (0.33 - b\ ,\ 0)$ and $p = 1$. A discharge $q_{in} = 0.18$ at the left boundary and a water height $h_{out} = 0.33$ at the right boundary are imposed. Figure 4 shows that the shock in the stationary state is captured sharply $(N = 100)$. Figure 5 presents the error measured in the L^2-norm. Owing to the presence of a shock, the observed order of convergence is $\dfrac{1}{2}$.

5.4. **Ritter solution** [Goutal, 1997]**.** This test illustrates the method of slope modification. The parameters are $g = 1$, $b(x) := 0$, $\ell = 40$, $T = 10$, $(\zeta_0\ ,\ q_0) = (\mathbb{1}_{]0,20[}\ ,\ 0)$ and $p = 1$. Figures 6 and 7 show the evolution of the approximate water height and the error in the L^2-norm. Figure 8 presents the error on the front position and the difference between the approximate and the exact mass of fluid for various mesh sizes. The exact front has a continuous advance unlike the approximate one which is always located at an extremity of a mesh interval. This fact explains the aspect of the error on the front. Moreover, the mass of fluid is not preserved exactly due to the use of a threshold for the dry areas but the difference is negligible (less than 10^{-5} for $N = 50$ and 10^{-6} for $N = 100$ at final time).

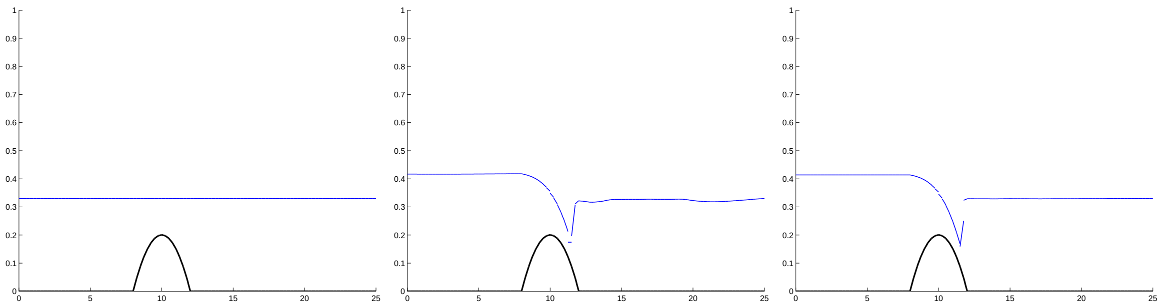

Figure 4: Approximate water height at times $t = 0,\ 25$ and 200 $(N = 100,\ p = 1)$

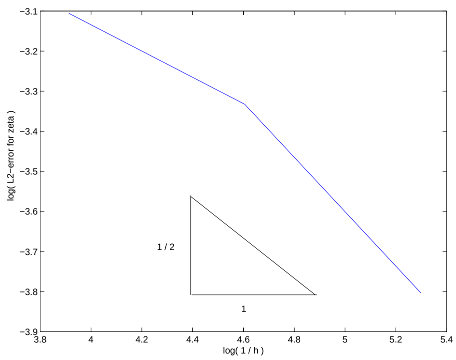

Figure 5: $\log(\|\zeta - \zeta_h\|_{0,]0,\ell[,t=T})$ versus $\log(N)$

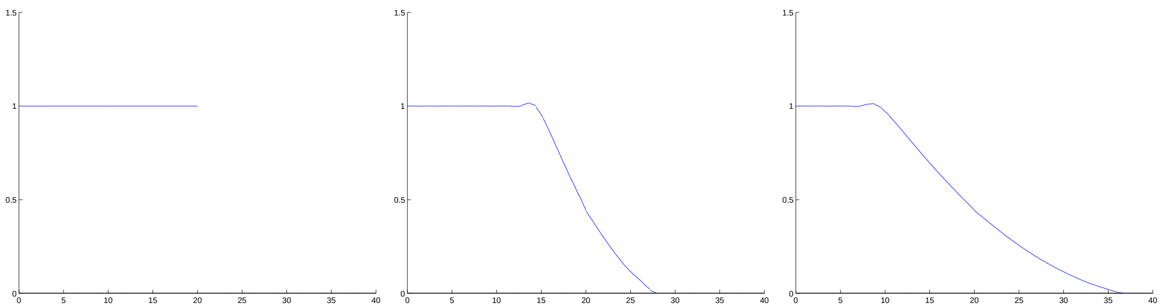

Figure 6: Approximate water height at times $t = 0,\ 5$ and 10 $(N = 100,\ p = 1)$

5.5. **Carrier-Geenspan** [Carrier, 1958, George, 2004]**.** Details about this test can be found in [Carrier, 1958, George, 2004]. Figure 9 illustrates the evolution of the approximate water height. Figure 10 presents the convergence rate in the L^2-norm and the error on the front position for various mesh sizes. These results are comparable to those obtained in [George, 2004].

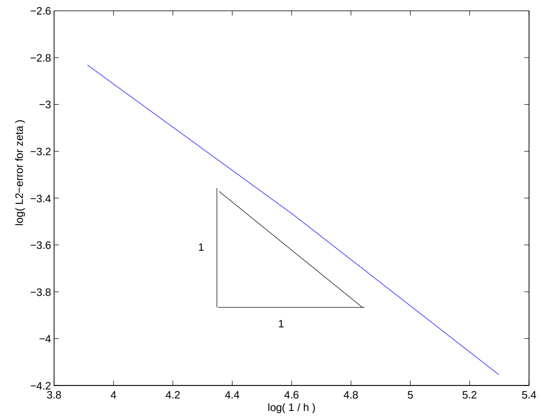

Figure 7: $\log(\|\zeta - \zeta_h\|_{0,]0,\ell[,t=T})$ versus $\log(N)$

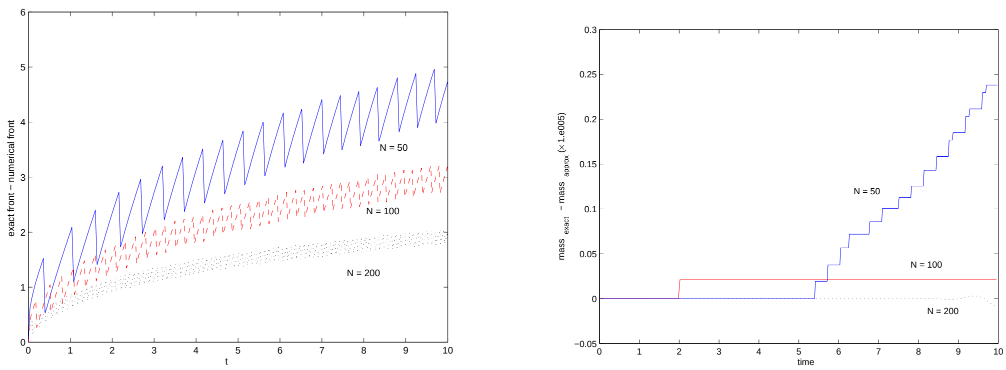

Figure 8: Errors on the front position (left) and on the mass of fluid (right) for various mesh sizes

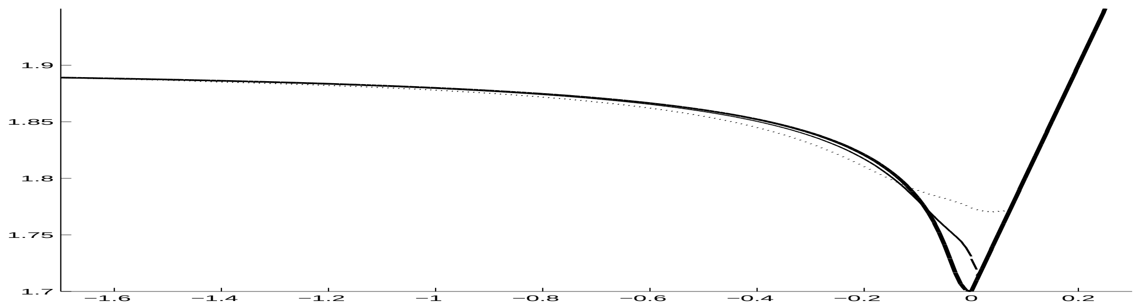

Figure 9: Approximate water height at times $t = 0$ (bold solid line), 0.2 (thin solid line) and 0.4 (dotted line) for $N = 100,\ p = 1$

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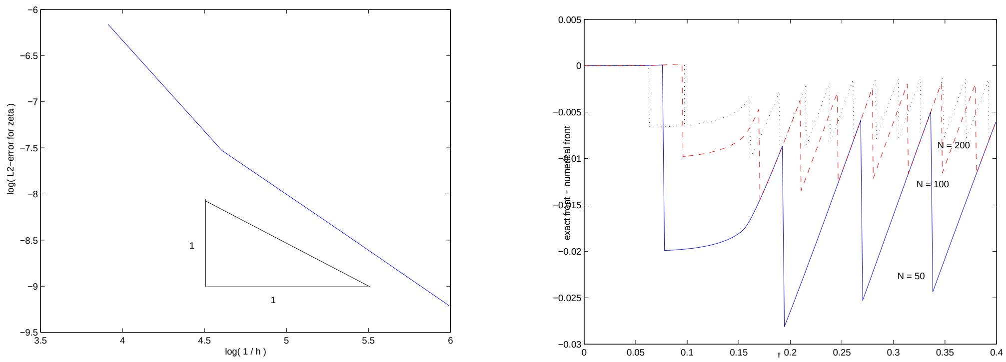

Figure 10: $\log(\|\zeta - \zeta_h\|_{0,]0,\ell[,t=T})$ versus $\log(N)$ (left) and error on the front position for different mesh sizes (right)

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