NUMERICAL SOLUTION OF THE TWO-DIMENSIONAL RICHARDS EQUATION VIA A TAYLOR-FRÉCHET ELLAM TECHNIQUE

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ABSTRACT

The Eulerian-Lagrangian Localized Adjoint Method (ELLAM) has been successfully employed in the solution of advection-dominated linear transport problems. Since the concept of adjoint operator only exists for linear equations, ELLAM is strictly speaking limited to linear problems. Some linearization strategies have been proposed, such as the one based on a Picard type of scheme. Nevertheless, these strategies do not work well for the solution of highly nonlinear problems, such as those governed by Richards' equation, that describes the flow of liquids in partially saturated soils. It is a well known fact that traditional Eulerian techniques, such as the Finite Difference or the Finite Element Method require the use of extremely fine meshes for the solution of Richards' equation, particularly in cases where sharp moisture content or pressure gradients propagate through initially dry soils. This paper presents the development of a technique based on the Taylor-Fréchet expansion of the nonlinear two-dimensional Richards operator, following the characteristics of the governing equation, properly cast in an advection-diffusion-reaction format. An operator is linearized and a rapidly convergent solution technique is generated. Several example that illustrate the application of the proposed procedure is presented. A comparison with conventional Eulerian methods and the Picard linearization procedure clearly show the superiority of the Taylor-Fréchet ELLAM technique.

Keywords : Unsaturated soils, Richards' equation, ELLAM, Taylor-Fréchet expansion.

1 INTRODUCTION

The motion of water in unsaturated soils is described by Richards' equation, which in two dimensions may be written as:

$$N(\psi) \equiv S(\psi)\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial z}\left(K(\psi)\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial x}\left(K(\psi)\frac{\partial \psi}{\partial x}\right) - K'\frac{\partial \psi}{\partial z} = 0; \tag{1}$$

$$\text{where}: S(\psi) = \frac{\partial \theta}{\partial \psi}; \qquad K' = \frac{\partial K}{\partial \psi}$$

$N(\cdot)$ represents the nonlinear Richards operator; $\theta(z,t)$ is the volumetric moisture content;

ψ(z,t), the pressure potential; $K(\psi)$, the hydraulic conductivity; $S(\psi)$, the specific capacity; x, the horizontal space coordinate; z, the vertical space coordinate positive oriented upwards, and t, time.

In view of the highly nonlinear nature of the relationships $\psi(\theta)$, $K(\psi)$ and $S(\psi)$, only very few analytical solutions of Eq. (1) are known for some idealized special cases. Thus, in order to have approximate solutions for cases of practical interest, it is necessary to resort to numerical techniques. Traditionally, Richards' equation has been solved by means of conventional Eulerian techniques, such as the Finite Difference Method or the Finite Element Method. The solutions obtained via these procedures are plagued with problems, such as spurious oscillations, excessive numerical diffusion and mass balance errors. To correct this undesirable performance, extremely fine space and time grids must be used, with the corresponding high computational expense.

The potential of the Eulerian-Lagrangian Localized Adjoint Method (ELLAM) has been demonstrated by various authors (Celia et al. 1990, Celia & Zisman 1990, Vag et al. 1990). ELLAM has been mainly employed to tackle linear problems with constant coefficients and in one dimension. In order to extend the application of ELLAM to nonlinear problems, a linearization procedure must be employed first, since strictly speaking, adjoint operators only exist for linear equations. The linearization methods that have been mostly used thus far are based on Picard-type strategies, where the nonlinear coefficients are evaluated at a previous time step, following the characteristics (Vag et al., 1990). Nevertheless, it has been observed that when the nonlinear terms are dominant, the convergence performance of the Picard method may be quite poor. Aldama and Arroyo (1998) have developed a linearization technique based on the Taylor-Fréchet expansion of the nonlinear Richards operator, properly cast in advection-diffusion-reaction form. The expansion is performed around values of the dependent variable at a previous time step, following the characteristics. In this expansion, terms of quadratic order in the difference between the solution evaluated at the current time value and the previous one, are neglected, thus generating a linear equation. The adjoint operator of this equation is then determined and ELLAM may be applied.

2 INITIAL AND BOUNDARY CONDITIONS

Let Eq. (1) be valid in the spatial domain $(x,z)\in\Omega$ and the time domain $t\in(0,T]$, and be subject to the following initial and boundary conditions:

$$\begin{aligned}&\psi(x,z,0)=g(x,z);\quad (x,z)\in\Omega\\ &\psi(x,z,t)=f(x,z,t) \text{ on } (x,z)\in\partial\Omega^1,\ t\in(0,T]\\ &-\left[(K(\psi)\mathbf{k}+K\nabla\psi)\right]\cdot\mathbf{n}=q(x,z,t) \text{ on } (x,z)\in\partial\Omega^2,\ t\in(0,T]\end{aligned}\tag{2}$$

where $\partial\Omega=\partial\Omega^1\cup\partial\Omega^2$ is the boundary of the domain $\Omega_{x,z}$, $\mathbf{n}$ is the unit vector normal to $\partial\Omega^2$ and $\mathbf{k}$ is the unit vector in the direction of z.

3 TAYLOR-FRÉCHET EXPANSION

Let

$$\psi(x,z,t) = \hat{\psi}(x,z,t) + \tilde{\psi}(x,z,t), \left\|\tilde{\psi}\right\| / \left\|\hat{\psi}\right\| << 1 \tag{3}$$

where $\hat{\psi}(x,z,t) = \psi(x-\xi, z-\zeta, t-\tau)$, $\xi = \int_{t-\tau}^{t} u\, d\tilde{\tau}$, $\zeta = \int_{t-\tau}^{t} w\, d\tilde{\tau}$; u and w respectively represent the components of the advective velocity vector corresponding to Richards' equation, properly cast in advective-diffusive-reactive form, as will be shown below. The term $\tilde{\psi}(x,z,t)$ represents a correction function. Following the strategy proposed by Aldama and Arroyo (2002), a Taylor-Fréchet expansion of Richards equation (1) may be employed to linearize it. Such expansion may be represented as: $N(\hat{\psi}+\tilde{\psi}) = N(\hat{\psi}) + dN(\hat{\psi}+\tilde{\psi}) + O(\left\|\tilde{\psi}\right\|^2) = 0$, where $dN(\tilde{\psi};\hat{\psi})$ is the first order Fréchet derivative of the nonlinear Richards operator $N(\cdot)$, which is a linear operator acting on $\tilde{\psi}$, and which has a nonlinear parametric dependence on $\hat{\psi}$.

It is readily shown that (Arroyo, 2005):

$$dN(\tilde{\psi};\hat{\psi}) = S(\hat{\psi})\frac{\partial\tilde{\psi}}{\partial t} - \frac{\partial}{\partial z}\left[\bar{K}\tilde{\psi} + K(\hat{\psi})\frac{\partial\tilde{\psi}}{\partial z}\right] - \frac{\partial}{\partial x}\left[\bar{L}\tilde{\psi} + K(\hat{\psi})\frac{\partial\tilde{\psi}}{\partial x}\right] + \bar{S}\tilde{\psi} \tag{4}$$

where:

$$\bar{L} = K'(\hat{\psi})\frac{\partial\hat{\psi}}{\partial x}, \quad \bar{K} = K'(\hat{\psi})\left(1 + \frac{\partial\hat{\psi}}{\partial z}\right) \text{ and} \tag{5}$$

Neglecting terms of $O(\left\|\tilde{\psi}\right\|^2)$ in the Taylor-Fréchet expansion of Richards equation, the following non-homogeneous equation, which is linear in $\tilde{\psi}$, results:

$$dN(\tilde{\psi};\hat{\psi}) = -N(\hat{\psi}) \tag{6}$$

Eq. (6), in expanded form, is

$$\begin{aligned} &S(\hat{\psi})\frac{\partial\tilde{\psi}}{\partial t} - \nabla\cdot\mathbf{v}\tilde{\psi} - \nabla\cdot\left[K(\hat{\psi})\nabla\tilde{\psi}\right] + \bar{S}\tilde{\psi} = \\ &-S(\hat{\psi})\frac{\partial\hat{\psi}}{\partial t} + \nabla\cdot\left[K(\hat{\psi})\nabla\hat{\psi}\right] + \frac{\partial K(\hat{\psi})}{\partial z} \end{aligned} \tag{7}$$

where $\mathbf{v} \equiv (\bar{L}, \bar{K})$. An iterative procedure may be implemented to tackle Eq. (7). Thus, the values of $\hat{\psi}$, $\tilde{\psi}$ and ψ may be updated by employing the algoritm proposed by Aldama and

Arroyo (2002). By using a Taylor series expansion, Arroyo (2005) has shown that even for the first iteration, a second order accurate scheme results, provided a time integration method with corresponding accuracy is employed. Since Eq. (7) is linear, ELLAM may be employed to solve it.

4 ELLAM FORMULATION

Following ELLAM, let the first weak form of Eq. (7) be considered:

$$\begin{aligned}&\int_0^T \int_\Omega \left\{ S(\psi)\frac{\partial \tilde{\psi}}{\partial t} - \nabla\cdot\mathbf{v}\tilde{\psi} - \nabla\cdot\left[K(\psi)\nabla\tilde{\psi}\right] + \bar{S}\tilde{\psi} \right\} w(x,z,t) = \\ &\int_0^T \int_\Omega \left\{ -S(\psi)\frac{\partial \psi}{\partial t} + \nabla\cdot\left[K(\psi)\nabla\psi + K\mathbf{k}\right] \right\} w(x,z,t)\end{aligned} \tag{8}$$

where $w(x,z,t)$ is a weight function that will be defined later on. Upon integration by parts, Eq. (8) is transformed into:

$$\begin{aligned}&\int_0^T \int_\Omega \left\{ S\frac{\partial(\tilde{\psi}w)}{\partial t} + (K\nabla\tilde{\psi})\cdot\nabla w - \nabla\cdot\left[(\mathbf{v}\tilde{\psi}w) + (K\nabla\tilde{\psi})w\right] \right. \\ &\left. -\tilde{\psi}\left[S\frac{\partial w}{\partial t} - \mathbf{v}\cdot\nabla w - \bar{S}w\right]\right\} d\Omega dt = \\ &\int_0^T \int_\Omega \left\{ -S\frac{\partial(\psi w)}{\partial t} - (K\nabla\psi + K\mathbf{k})\cdot\nabla w + \nabla\cdot\left[K(\psi)\mathbf{k}w + (K\nabla\psi)w\right] - \bar{S}\psi w - \mathbf{v}\psi\cdot\nabla w \right. \\ &\left. -\psi\left[S\frac{\partial w}{\partial t} - \mathbf{v}\cdot\nabla w - \bar{S}w\right]\right\} d\Omega dt\end{aligned} \tag{9}$$

In the ELLAM formulation, $w(x,z,t)$ is chosen in such a way that the last terms in both sides of Eq. (9) vanish. On the other hand, Green´s formula transforms the third terms in both sides into a boundary integrals. The fourth and the fifth term on the right hand side are handled via algebraic manipulations. Thus, the following result is obtained:

$$\begin{aligned}&\int_0^T \int_\Omega \left\{ S\frac{\partial(\tilde{\psi}w)}{\partial t} + (K\nabla\tilde{\psi})\cdot\nabla w \right\} d\Omega dt - \int_0^T \int_{\partial\Omega} \left[(\mathbf{v}\tilde{\psi}) + (K\nabla\tilde{\psi})\right]\cdot\mathbf{n}\, w dS dt = \\ &\int_0^T \int_\Omega \left\{ -S\frac{\partial(\psi w)}{\partial t} - (K\nabla\psi + K\mathbf{k})\bullet\nabla w - \bar{S}\psi w - \mathbf{v}\psi\cdot\nabla w \right\} d\Omega dt - \int_0^T \int_{\partial\Omega} \left[(K(\psi)\mathbf{k} + K\nabla\psi)\right]\cdot\mathbf{n} w dS dt\end{aligned} \tag{10}$$

where dS represents a boundary arc differential and $\mathbf{n}$ is a unit vector normal to the boundary $\partial\Omega$.

As was implied earlier, the weight function must satisfy the following equation

$$S\frac{\partial w}{\partial t} - \mathbf{v}\cdot\nabla w - \bar{S}w = 0 \qquad (11)$$

Employing Eq. (5) in Eq. (11) results in:

$$\frac{\partial w}{\partial t} - \frac{\bar{L}}{S}\frac{\partial w}{\partial x} - \frac{\bar{K}}{S}\frac{\partial w}{\partial z} - \frac{\bar{S}}{S}w = 0 \qquad (12)$$

By setting $u = \bar{L}/S(\psi)$, $w = \bar{K}/S(\psi)$ and $\gamma = \bar{S}/S(\psi)$, Eq. (14) may be written as:

$$\frac{\partial w}{\partial t} - u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - \gamma w = 0 \qquad (13)$$

which is an advection-diffusion-reaction equation. Hence, u is the x component of the advective velocity and w, the y component.

There is an infinite number of functions that satisfy Eq. (13), but it is convenient to chose the most simple one. In practice, the simplest functions are those that are piecewise linear or piecewise constant in space. Aldama and Arroyo (2000) have demonstrated that the propagation characteristics of both are similar. Accordingly, the space piecewise constant choice has been employed in this work, due to its simplicity. In addition, the weight function is considered to be exponentially varying in time (for details, see Arroyo, 2005).

Once the weight function has been selected, the spatial domain and the time domain are discretized for the solution of Eq. (10). The important consideration to take into account in this regard is that, according to the Lagrangian component of the method, spatial defined in at the current computational time, are projected backwards, following the characteristics, by means of the corresponding values of the advective velocity components.

5 NUMERICAL EXAMPLE

Let the infiltration of water in a 5×5 meters two-dimensional domain of an initially very dry soil be considered. The initial condition corresponds to a constant value of the pressure potential equal to -1.0 m. The surface boundary condition (z=5 m) is of the Neumann kind, with a Darcy flux equal to: 0.01 m/h at a segment of 1 m length, located at the center of the surface, and zero elsewhere. At the bottom of the domain (z=0), a Dirichlet boundary condition, with the pressure potential set equal to -1.0 is specified. A zero gradient boundary condition is considered at the lateral boundaries of the domain.

The hydrodynamic characteristics of the soil are obtained through van Genuchten relations, with the following parameters: residual moisture, $\theta_r = 0.15$; saturation moisture $\theta_s = 0.38$; empirical parameter, $\psi_s = -1.2\text{ m}$; pore size distribution index, n=4; saturation hydraulic conductivity, $K_s = 0.0004\text{ m/h}$, and specific storage parameter, $S_s = 0.0001$.

A surrogate "exact" solution was obtained by employing a finite difference approximation of Richards' equation on a very fine grid. The discretization parameters employed were $\Delta x = \Delta z = 0.125\text{ m}$ and $\Delta t = 0.25\text{ h}$. These values were determined to be adequate since the solution obtained by employing them, was not significantly different from that corresponding to halving them.

The "exact" solution corresponding to t=1000 h is shown in Figure 1. The solution obtained with ELLAM and the Taylor-Fréchet linearization procedure proposed here, in two iterations, is shown in Figure 2. In this case the discretization parameters were: $\Delta x = \Delta z = 0.125$ m and $\Delta t = 25$ h, i.e., a time step a hundred times larger to the one employed in the "exact solution" was employed, with the corresponding savings in computational effort. As may be seen, the two solutions are practically indistinguishable.

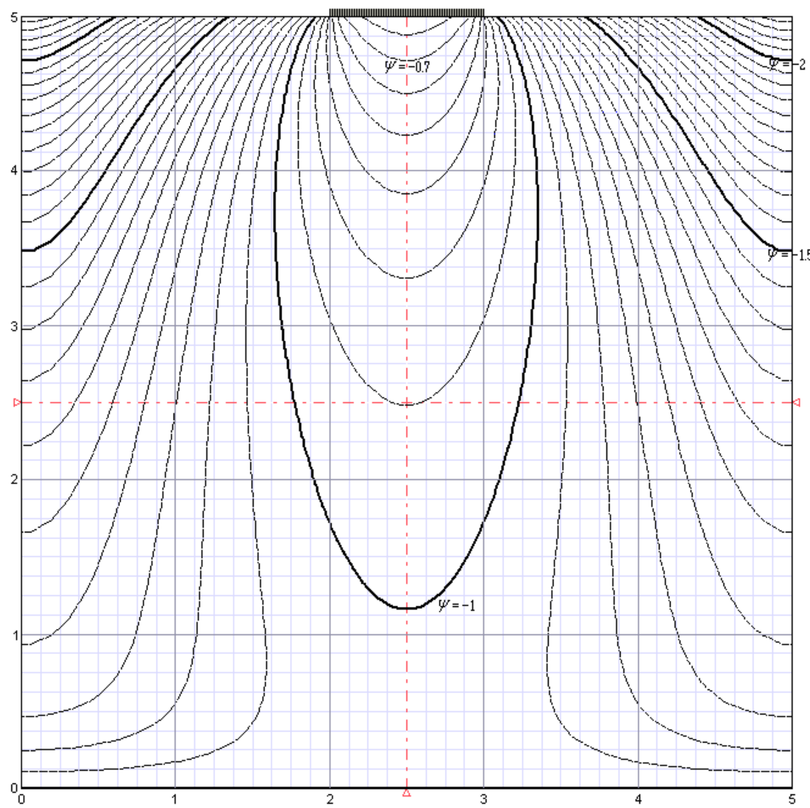

Figure 1. "Exact" solution.

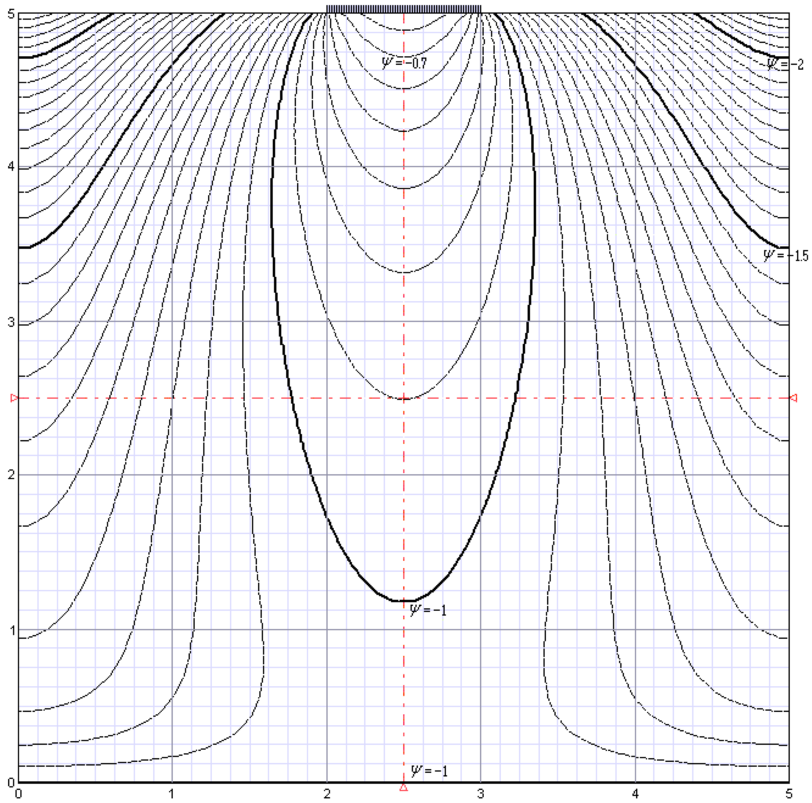

Figure 2. Taylor-Fréchet ELLAM solution.

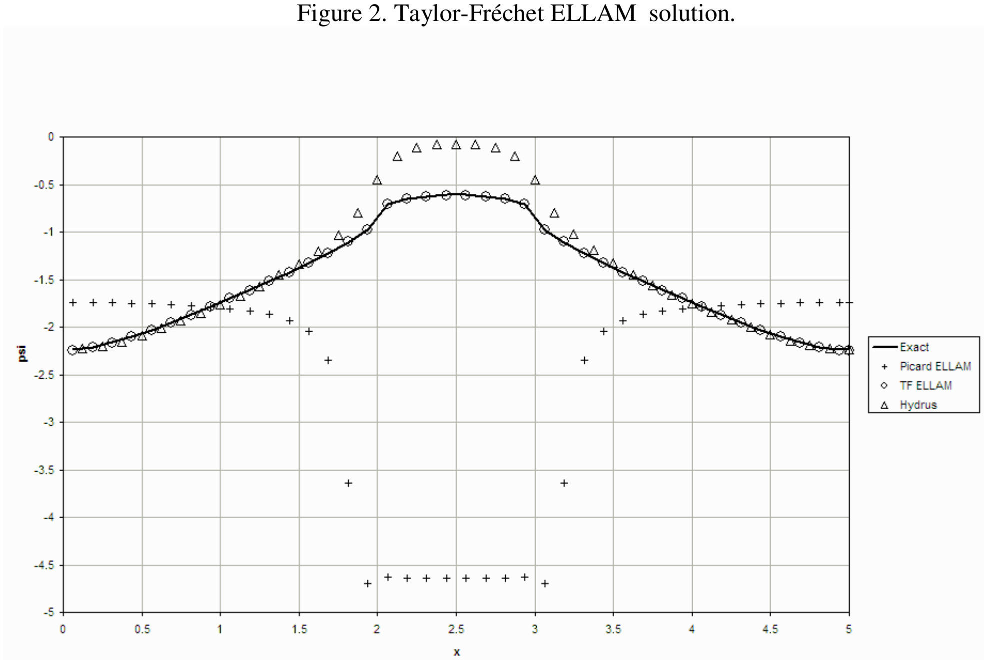

Figure 3. Comparison of results for a horizontal cut at z=5 and t=1000.

Additional simulations were run by employing a conventional Finite Element discretization and ELLAM with a Picard linearization procedure. In both cases, the discretizations parameters $\Delta x = \Delta z = 0.125$ m and $\Delta t = 25$ h , and two iterations, were used. A comparison corresponding to a horizontal cut at *z*= 5 m and *t*=1000 h is shown in Figure 3.

As may be observed, the results corresponding to the "exact" solution and Taylor-Fréchet ELLAM are practically indistinguishable. The Finite Element solution behaves poorly at the center of the domain, whereas Picard ELLAM behaves very poorly everywhere (for more details, see Arroyo, 2005).

6 CONCLUSIONS

A method for the linealization of the two-dimensional Richards equation that may be very efficiently used in conjunction with ELLAM is developed. The technique rests on the employment of a Taylor-Fréchet expansion of the nonlinear Richards operator, in the direction of the characteristics. A difficult numerical test is considered to evaluate the performance of the procedure, consisting of the propagation of a wet front in an initially dry soil. It must be said that traditional numerical methods face many difficulties in tackling this type of problems, since hydraulic conductivity gradients are very high, thus making them strongly dominated by advection. The performance of the Taylor-Fréchet ELLAM scheme is excellent, while traditional Finite Element and Picard ELLAM generate poor results.

REFERENCES

Aldama, A. and V. Arroyo, 1998. "An Eulerian-Lagrangian localized adjoint method for the nonlinear advection-diffusion-reaction equation", Proc. XII Int. Conf. Computational Methods in Water Resources, 2, 569-576.

Aldama, A. and V. Arroyo, 2000. "Propagation properties of Eulerian-Lagrangian localized adjoint methods", Proc. XIII Int. Conf. Computational Methods in Water Resources, 2, 597-601.

Aldama, A. and V. Arroyo, 2002. Numerical simulation of unsaturated flow via ELLAM, *Developments in Water Science*, 47 (2), 1027-1034.

Arroyo, V. M., 1994. "Modelo unidimensional de simulación numérica para drenaje agrícola", Tesis de maestría en Ingeniería Hidráulica. Facultad de Ingeniería. UNAM. pp. 155.

Arroyo, V. and A. Aldama, , 2002. "Simulación numérica del flujo del agua en suelos no saturados mediante ELLAM", II Congreso Internacional de Métodos Numéricos en Ingeniería y Ciencias Aplicadas. Guanajuato, Gto. México.1:805-814.

Arroyo, V. M., 2005. "Modelación bidimensional de flujo y transporte no lineales en medios porosos aplicando ELLAM", Tesis Doctoral en Ingeniería Hidráulica. Facultad de Ingeniería. UNAM. pp. 333.

Celia, M., T. Russell, I. Herrera, and R. Ewing, 1990. "An Eulerian-Lagrangian localized adjoint method for the advection-diffusion equation", *Adv. Water Res.*, 13, 187-206.

Celia, M. y S. Zisman, 1990. "An Eulerian-Lagrangian localized adjoint method for reactive transport in groundwater", Comput. Meth. Water Res.1:289-296.

Vag, E., J. Wang, y D. Helge, 1990. "Eulerian-Lagrangian localized adjoint methods for system of nonlinear advective-diffusive-reactive transport equations", *Adv. Water Res.*,19, 297-315.